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Weighted intriguing sets of finite generalised quadrangles

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Abstract We construct and analyse interesting integer valued functions on the points of a generalised quadrangle which lie in the orthogonal complement of a principal eigenspace of the collinearity relation. These functions generalise the *intriguing sets* introduced by Bamberg, Law and Penttila [3], and they provide the extra machinery to give new proofs of old results and to establish new insight into the existence of certain configurations of generalised quadrangles. In particular, we give a *geometric* characterisation of Payne's tight sets, we give a new proof of Thas' result that an m-ovoid of a generalised quadrangle of order (s, s^2) is a hemisystem, and we give a bound on the values of m for which it is possible for an m-ovoid of the four dimensional Hermitian variety to exist.

Keywords Generalised quadrangle, hemisystem, m-ovoids, strongly regular graph

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1 Introduction

If one looks at the point graph of a generalised quadrangle $\mathcal{G}$, one will find a strongly regular graph. The associated Bose-Mesner algebra of this graph has an orthogonal decomposition into three eigenspaces of the adjacency matrix, one of which is the one-dimensional subspace consisting of the constant vectors (c.f., [5]). The other two eigenspaces are known as the *principal eigenspaces* of the point graph of $\mathcal{G}$. If $\mathcal{I}$ is a set of points of $\mathcal{G}$ such that its characteristic function $\mathcal{X}_{\mathcal{I}}$ is contained in the orthogonal complement of a principal eigenspace E, then there are constants h_1 and h_2 such that the number of points of $\mathcal{I}$ collinear to an arbitrary point p is h_1 if p lies in $\mathcal{I}$, and h_2 if p resides outside of $\mathcal{I}$. Such sets were termed *intriguing sets* in [3], and the points lying on a line of a generalised quadrangle is such an object where h_1 is the number of points on a line and $h_2 = 1$. Eisfeld [7] asks whether intriguing sets have a natural geometric interpretation, and it is shown in [3] that the intriguing sets of a generalised quadrangle are precisely the m-ovoids and tight sets introduced by J. A. Thas [14] and S. E. Payne [11] respectively. An m-ovoid and an i-tight set intersect in mi points [3, Theorem 4.3], and from this observation, one can prove or reprove interesting results about generalised quadrangles. We endeavour to extend this principle

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further, by not only considering characteristic functions $\mathcal{X}_S$ in the orthogonal complement of a principal eigenspace, but other maps in these subspaces. One of the recurring themes of this paper is the use of "weighted" intriguing sets and their combinatorial properties.

In 1965, Segre [13] showed that if $\mathcal{M}$ is an m-ovoid of $\mathsf{Q}^-(5,q)$ then $m = (q+1)/2$, that is, $\mathcal{M}$ is a *hemisystem*. J. A. Thas [14] generalised this result in 1989 by proving the following, as a corollary of a much more general theorem on m-ovoids:

Theorem 1.1 ([14]) *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2). If S is an m-ovoid of $\mathcal{G}$ with $0 < m < s+1$, then S is a hemisystem of $\mathcal{G}$.*

Recently, Vanhove [15, Theorem 3] has generalised this result to m-ovoids of regular near polygons with a certain distance parameter. We also give an alternate proof of Theorem 1.1 in the same vein as Vanhove's proof in Section 8, with the view to understanding why generalised quadrangles with parameters (s, s^2) have such a restriction on the sizes of their m-ovoids. We show that the central reason for this is that for every pair of non-collinear points x, y the hyperbolic line $|\{x,y\}^{\perp\perp}|$ has size $s^2/t+1$ and that for each z not in the closure of x and y (see §4.7 for a definition), we have $|\{x,y,z\}^{\perp}| = t/s+1$. Notice that this condition is weaker than every triad having a constant number of centres, as there are other generalised quadrangles with different parameters that satisfy our condition, namely $\mathsf{H}(4,q^2)$ and $\mathsf{W}(3,q)$. Indeed, we also apply our result to $\mathsf{H}(4,q^2)$, in Section 9.

It was proved in [3, Theorem 7.1] that an m-ovoid of $\mathsf{H}(4,q^2)$ must satisfy $m^2(q^2-1)^2+3m(q^2-1)-q^5 \geqslant 0$, and it is not yet known whether a (nontrivial) m-ovoid of $\mathsf{H}(4,q^2)$ exists. In this paper we give an alternative proof of the aforementioned bound on m, using a more insightful method.

Apart from these direct applications of our analysis of weighted intriguing sets, we also provide background theory (Sections 2 and 3) and theoretical material (Sections 4, 5 and 6) to support our results. In particular, we give a geometric characterisation of weighted tight sets in terms of their intersection with *weighted cones*, in a similar sense to the fact that a weighted m-ovoid is defined by its constant intersection property with respect to lines (Lemma 4.3). We also give geometric descriptions of the eigenspaces of the collinearity relation of a generalised quadrangle, and similarly for a partial quadrangle arising from the derivation of a generalised quadrangle of order (s, s^2) (Sections 5 and 6). These results generalise some of the known theory established in [1]. In Section 7, we prove that a Payne derived generalised quadrangle arising from $\mathsf{W}(3,q)$, q even, can be partitioned into ovoids and hence has m-ovoids for every possible m. The last section of this paper, Section 10, shows that one can construct m-ovoids of the dual classical generalised quadrangle $\mathsf{DH}(4,q^2)$ for certain values of m, thus solving a problem stated in [3, Question 9.2].

2 Some basic algebraic combinatorics

Let A be the adjacency matrix of a strongly regular graph Γ (which is not null or complete) with parameters (n,k,λ,μ). Then A has eigenvalues k, e^+ and e^- with multiplicities 1, f^+ and f^- respectively [5, §1.3]:

$$e^+ = \tfrac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} \qquad f^+ = \tfrac{k(e^-+1)(k-e^-)}{(k+e^+e^-)(e^--e^+)}$$
$$e^- = \tfrac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} \qquad f^- = \tfrac{k(e^++1)(k-e^+)}{(k+e^+e^-)(e^+-e^-)}.$$

Notice that $e^+ > 0$ and $e^- < 0$.

Consider the vector space $\mathbb{C}(V\Gamma)$ of all functions from the set of vertices $V\Gamma$ to the complexes. Then A induces an endomorphism of this vector space:

$$f^A := v \mapsto \sum_{A(v,w)=1} f(w).$$

Since A is symmetric, the three eigenspaces V^0, V^+ and V^-, with associated eigenvalues k, e^+ and e^- respectively, form a direct decomposition of $\mathbb{C}(V\Gamma)$:

$$\mathbb{C}(V\Gamma) = V^0 \oplus V^+ \oplus V^-.$$

Let $\mathbf{j}$ be the constant map with value 1, that is, $\mathbf{j} = \mathcal{X}_{V\Gamma}$, and notice that $V^0 = \langle \mathbf{j} \rangle$. One of the themes of this paper is the study of subsets of geometries which have their characteristic function in the orthogonal complement of one of the above direct summands.

Lemma 2.1 *Let f be a function in $\mathbb{C}(V\varGamma)$. Then the following statements are equivalent.*

(a) $f \in (V^\epsilon)^\perp$ *where* $\epsilon \in \{-,+\}$*;*
(b) There exists $b \in \mathbb{C}$ *such that* $Af = e^{-\epsilon}f + b\mathbf{j}$*;*
(c) There exists $a \in \mathbb{R}$ *and* $b \in \mathbb{C}$ *such that* $Af = af + b\mathbf{j}$*, with* $a > 0$ *and* $a \neq k$ *if* $\epsilon = -$ *and with* $a < 0$ *if* $\epsilon = +$*;*
(d) There exists $b \in \mathbb{C}$ *such that* $(e^{-\epsilon} - k)f + b\mathbf{j} \in V^{-\epsilon}$*.*

Proof Assume (a). Thus $f \in V^0 + V^{-\epsilon}$, so $f = t\mathbf{j} + v$ where $t \in \mathbb{C}$ and $v \in V^{-\epsilon}$. Thus $Af = tk\mathbf{j} + e^{-\epsilon}v = tk\mathbf{j} + e^{-\epsilon}(f - t\mathbf{j}) = e^{-\epsilon}f + t(k - e^{-\epsilon})\mathbf{j}$. Taking $b = t(k - e^{-\epsilon})$, (b) holds. Obviously (b) implies (c). Assume (c), so that $Af = af + b\mathbf{j}$. Then it is easy to check that $(a - k)f + b\mathbf{j}$ is an eigenvector corresponding to the eigenvalue a. As $a \neq k$, it means either $a = e^+$ if $\epsilon = -$, or $a = e^-$ if $\epsilon = +$. Thus (d) holds. Assume (d), so that $(e^{-\epsilon} - k)f + b\mathbf{j} = v$ where $v \in V^{-\epsilon}$. Thus $f = (-b\mathbf{j} + v)/(e^{-\epsilon} - k) \in V^0 + V^{-\epsilon} = (V^\epsilon)^\perp$, so (a) holds. □

We can equip $\mathbb{C}(V\varGamma)$ with a natural inner product, namely

$$f \cdot g := \sum_{v \in V\varGamma} f(v)g(v).$$

Let S be a proper nonempty subset of vertices of $\varGamma$ equipped with integral weights (possibly negative). Then the characteristic vector $\mathcal{X}_S$ is just a vector with integer entries. We say that S is a *weighted intriguing set* if $\mathcal{X}_S$ satisfies the equivalent conditions of Lemma 2.1. Note that the condition $a \neq k$ in (c) is automatically satisfied if $\mathcal{X}_S \in \{0,1\}^{V\varGamma}$, i.e., if S is an *unweighted* set.

Notation:

The *size* $|f|$ of an element f of $\mathbb{C}(V\varGamma)$ is $f \cdot \mathbf{j}$. In a point/line incidence geometry, we will use the $\perp$ symbol to describe the set $S^\perp$ of points which are collinear with every element of a set S of points. To be consistent, we will always assume that p lies in $p^\perp$ when we are working with a geometry, and use the notation $p^\sim$ for $p^\perp \backslash \{p\}$ to simulate the adjacency relation in a graph. We will often view a function $f \in \mathbb{C}\varOmega$, where $\varOmega$ is finite, as a $|\varOmega|$-tuple in $\mathbb{C}^{|\varOmega|}$.

We will exploit the following simple fact many times in this paper.

Lemma 2.2 *Let f, g be two elements of $\mathbb{C}(V\varGamma)$, where $f \in (V^+)^\perp$ and $g \in (V^-)^\perp$. Then*

$$f \cdot g = \frac{|f||g|}{|V\varGamma|}.$$

Proof By Lemma 2.1, there exist $b^+, b^- \in \mathbb{C}$ such that $(e^- - k)f + b^-\mathbf{j} \in V^-$ and $(e^+ - k)g + b^+\mathbf{j} \in V^+$. Since these elements are orthogonal, and $\mathbf{j}$ is orthogonal to both we have:

$$(k - e^-)(f \cdot \mathbf{j}) = b^-(\mathbf{j} \cdot \mathbf{j}), \quad (k - e^+)(g \cdot \mathbf{j}) = b^+(\mathbf{j} \cdot \mathbf{j})$$

and

$$\begin{aligned}(e^- - k)(e^+ - k)(f \cdot g) &= -b^-(e^+ - k)(g \cdot \mathbf{j}) - b^+(e^- - k)(f \cdot \mathbf{j}) - b^-b^+(\mathbf{j} \cdot \mathbf{j}) \\ &= b^-b^+(\mathbf{j} \cdot \mathbf{j}) + b^+b^-(\mathbf{j} \cdot \mathbf{j}) - b^-b^+(\mathbf{j} \cdot \mathbf{j}) \\ &= b^-b^+(\mathbf{j} \cdot \mathbf{j}).\end{aligned}$$

Therefore, $\frac{b^-b^+(\mathbf{j}\cdot\mathbf{j})^2}{(f\cdot\mathbf{j})(g\cdot\mathbf{j})}(f \cdot g) = b^-b^+(\mathbf{j} \cdot \mathbf{j})$ and hence $f \cdot g = \frac{|f||g|}{\mathbf{j}\cdot\mathbf{j}}$. □

3 Generalised quadrangles

Much of the material in this section can be found in [5] and [12]. A finite generalised quadrangle is an incidence structure of points and lines such that: every pair of different points lie on at most one line, there are constants s and t such that every line contains exactly $s+1$ points and every point lies on $t+1$ lines, and given a point p and a line ℓ which are not incident, there is a unique line on p sharing a point with ℓ. The last of these conditions is sometimes known as the *GQ-axiom*. With the parameters above, we say that a generalised quadrangle has *order* (s,t). It has $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines. The dual structure is also a generalised quadrangle, but of order (t,s). We often identify a line with the set of points lying on that line. The point graph of a generalised quadrangle $\mathcal{G}$ of order (s,t) has as vertices the points of $\mathcal{G}$, and two vertices are adjacent if they are collinear and distinct. This graph is strongly regular with eigenvalues and multiplicities listed below [5, pp. 203]:

Eigenvalue	Multiplicity
$s(t+1)$	1
$e^+ = s-1$	$\frac{st(s+1)(t+1)}{s+t}$
$e^- = -t-1$	$\frac{s^2(st+1)}{s+t}$

We will always consider *thick* generalised quadrangles, where s and t are both greater than 1. The Higman inequality stipulates that for $s>1$, the conditions $s \leqslant t^2$ and $t \leqslant s^2$ hold for any generalised quadrangle of order (s,t) (see [12, §1.2.5]). It is useful to also have a list of the *classical* generalised quadrangles for future reference:

Name	$\mathsf{W}(3,q)$	$\mathsf{Q}(4,q)$	$\mathsf{Q}^-(5,q)$	$\mathsf{H}(3,q^2)$	$\mathsf{H}(4,q^2)$
Order	(q,q)	(q,q)	(q,q^2)	(q^2,q)	(q^2,q^3)

Table 1 The classical generalised quadrangles and their parameters. The first and second are dual, and the third and fourth are dual.

Let $\mathcal{G}$ be a generalised quadrangle, with point-set $\mathcal{P}$ and line-set $\mathcal{L}$. Let A be the adjacency matrix of the point graph of $\mathcal{G}$. Recall that $x^\perp$ denotes the cone with vertex x, that is, the point-set consisting of the points on all lines through a fixed point x, including x, and $x^\sim$ denotes the same set of points but excluding x. Since the set of all $\mathcal{X}_x$, $x \in \mathcal{P}$, forms an orthonormal basis of $\mathbb{C}\mathcal{P}$, for each $f \in \mathbb{C}\mathcal{P}$ we can write:

$$f = \sum_{x\in\mathcal{P}} (\mathcal{X}_x \cdot f)\mathcal{X}_x, \tag{1}$$

$$Af = \sum_{x\in\mathcal{P}} (\mathcal{X}_{x^\sim} \cdot f)\mathcal{X}_x = \sum_{x\in\mathcal{P}} \left(\sum_{\ell \ni x} \mathcal{X}_{\ell\setminus\{x\}} \cdot f \right) \mathcal{X}_x. \tag{2}$$

4 Some examples of (weighted) intriguing sets of generalised quadrangles

Consider a generalised quadrangle $\mathcal{G}$ of order (s,t). Below we give some simple examples of weighted intriguing sets of generalised quadrangles that we use throughout this paper.

4.1 Lines

We identify a line ℓ with the set of points of $\mathcal{G}$ lying on ℓ. Then by the definition of a generalised quadrangle, we have

$$|p^\sim \cap \ell| = \begin{cases} s & \text{if } p \in \ell \\ 1 & \text{if } p \notin \ell \end{cases}$$

and hence

$$A\mathcal{X}_\ell = (s-1)\mathcal{X}_\ell + \mathbf{j}.$$

Therefore, $\mathcal{X}_\ell \in (V^-)^\perp$ by Lemma 2.1. This is an example of a 1-*tight set*.

4.2 Tight sets

A set of points $\mathcal{T}$ of a generalised quadrangle of order (s,t) is an *i-tight set* if for every point p in $\mathcal{T}$, there are $s+i$ points of $\mathcal{T}$ collinear with p, and for every point p not in $\mathcal{T}$, there are i points of $\mathcal{T}$ collinear with p (see [11]). So

$$A\mathcal{X}_{\mathcal{T}} = (s-1)\mathcal{X}_{\mathcal{T}} + i\mathbf{j}$$

and therefore, $\mathcal{X}_{\mathcal{T}} \in (V^-)^\perp$ by Lemma 2.1. We say that a set of points is *tight* if it is i-tight for some i.

4.3 m-Ovoids

An *m-ovoid* $\mathcal{M}$ of $\mathcal{G}$ is a set of points such that every line meets $\mathcal{M}$ in m points (see [14]). It is not difficult to see that

$$A\mathcal{X}_{\mathcal{M}} = -(t+1)\mathcal{X}_{\mathcal{M}} + m(t+1)\mathbf{j}$$

and hence $\mathcal{X}_{\mathcal{M}} \in (V^+)^\perp$. Moreover, we have the following result:

Lemma 4.1 ([3, Theorem 4.1]) *Let S be a set of points of $\mathcal{G}$ such that $\mathcal{X}_S \in (V^\epsilon)^\perp$ (where $\epsilon \in \{-,+\}$). Then S is an m-ovoid or a tight set.*

The 1-ovoids are often called *ovoids* in the literature.

4.4 Weighted m-ovoids

We say that a weighted intriguing set S is a *weighted m-ovoid* if $\mathcal{X}_S \in (V^+)^\perp$, where the number m arises from the geometric property: $\mathcal{X}_S \cdot \mathcal{X}_\ell = m$ for all lines ℓ. Indeed we have the following more general result:

Lemma 4.2 *Let $f \in \mathbb{C}\mathcal{P}$. Then $f \in (V^+)^\perp$ if and only if there exists a constant $m \in \mathbb{C}$ such that $\mathcal{X}_\ell \cdot f = m$ for any line $\ell \in \mathcal{L}$. In that case*

$$\begin{aligned} &Af = -(t+1)f + m(t+1)\mathbf{j},\\ &(s+1)f - m\mathbf{j} \in V^-,\\ &f \cdot \mathbf{j} = m(st+1). \end{aligned}$$

Proof Suppose first that we have a constant $m \in \mathbb{C}$ such that $\mathcal{X}_\ell \cdot f = m$ for any line $\ell \in \mathcal{L}$. Then, using (1) and (2), we obtain

$$\begin{aligned} Af &= \sum_{x\in\mathcal{P}} \Big(\sum_{\ell\ni x} \mathcal{X}_{\ell\setminus\{x\}} \cdot f\Big)\mathcal{X}_x \\ &= \sum_{x\in\mathcal{P}} \Big(\sum_{\ell\ni x} (m - \mathcal{X}_x \cdot f)\Big)\mathcal{X}_x \\ &= \sum_{x\in\mathcal{P}} (m(t+1) - (t+1)\mathcal{X}_x \cdot f)\mathcal{X}_x \\ &= m(t+1)\sum_{x\in\mathcal{P}} \mathcal{X}_x - (t+1)\sum_{x\in\mathcal{P}} (\mathcal{X}_x \cdot f)\mathcal{X}_x \\ &= m(t+1)\mathbf{j} - (t+1)f. \end{aligned}$$

Since $-(t+1) < 0$, by Lemma 2.1, $f \in (V^+)^\perp$.

Now assume $f \in (V^+)^\perp$. Let ℓ be a line. As seen in §4.1, $\mathcal{X}_\ell \in (V^-)^\perp$ Hence by Lemma 2.2, $\mathcal{X}_\ell \cdot f = \frac{|f|(s+1)}{(s+1)(st+1)} = \frac{|f|}{st+1}$, which does not depend on the choice of ℓ, so we take $m = \frac{|f|}{st+1}$. By Lemma 2.1, we also have that $(-t-1-k)f + m(t+1)\mathbf{j} = -(t+1)((s+1)f - m\mathbf{j}) \in V^-$. □

4.5 Weighted cones

Let x be a point of $\mathcal{G}$. Let C_x be the set of all points collinear with x, and give every point of C_x the weight 1, except the point x, which will have weight $-s+1$. In other words $\mathcal{X}_{C_x} = (-s+1)\mathcal{X}_x + \mathcal{X}_{x^\sim}$. Then C_x is a weighted 1-ovoid by Lemma 4.2, as clearly $\mathcal{X}_\ell \cdot \mathcal{X}_{C_x} = 1$ for any line ℓ.

4.6 Weighted tight sets

A weighted tight set S is defined in much the same way as a weighted m-ovoid: $\mathcal{X}_S \in (V^-)^\perp$. By Lemma 2.1, a weighted tight set satisfies $A\mathcal{X}_S = (s-1)\mathcal{X}_S + b\mathbf{j}$ for some integer b, and we then call it a *weighted b-tight set*. A geometric characterisation of weighted tight sets follows from the following general result (weighted cones are described in §4.5):

Lemma 4.3 *Let $f \in \mathbb{C}\mathcal{P}$. Then $f \in (V^-)^\perp$ if and only if there exists a constant $b \in \mathbb{C}$ such that $\mathcal{X}_{C_x} \cdot f = b$ for any weighted cone C_x. In that case*

$$\begin{aligned}&Af = (s-1)f + b\mathbf{j},\\ &-(1+st)f - b\mathbf{j} \in V^+,\\ &f \cdot \mathbf{j} = b(s+1).\end{aligned}$$

Proof Suppose first that we have a constant $b \in \mathbb{C}$ such that $\mathcal{X}_{C_x} \cdot f = b$ for any weighted cone C_x. Then, using (1) and (2), we obtain

$$\begin{aligned}Af &= \sum_{x\in\mathcal{P}} (\mathcal{X}_{x^\sim} \cdot f)\mathcal{X}_x\\ &= \sum_{x\in\mathcal{P}} ((\mathcal{X}_{C_x} + (s-1)\mathcal{X}_x) \cdot f)\mathcal{X}_x\\ &= \sum_{x\in\mathcal{P}} (\mathcal{X}_{C_x} \cdot f)\mathcal{X}_x + (s-1)\sum_{x\in\mathcal{P}} (\mathcal{X}_x \cdot f)\mathcal{X}_x\\ &= b\mathbf{j} + (s-1)f\end{aligned}$$

Since $(s-1) > 0$, by Lemma 2.1, $f \in (V^-)^\perp$.

Now assume $f \in (V^-)^\perp$ and let $b = \frac{|f|}{s+1}$. Let C_x be a weighted cone. We can easily compute that $|C_x| = (-s+1) + s(t+1) = 1 + st$. As seen in §4.5, $\mathcal{X}_{C_x} \in (V^+)^\perp$. Hence by Lemma 2.2, $\mathcal{X}_{C_x} \cdot f = \frac{|f|(st+1)}{(s+1)(st+1)} = b$, which does not depend on the choice of C_x. By Lemma 2.1, we also have that $(s-1-k)f + b\mathbf{j} = -(1+st)f - b\mathbf{j} \in V^+$. □

4.7 Linear combinations of a hyperbolic line and its perp

Here we generalise an example of a weighted tight-set given in the Appendix of [2]. Let x and y be two non-collinear points. It follows easily from the GQ-axiom that $|\{x,y\}^\perp| = t+1$ and that

$$2 \leqslant |\{x,y\}^{\perp\perp}| \leqslant t+1.$$

We call the set of points of $\{x,y\}^{\perp\perp}$ the *hyperbolic line* on x and y (notice it contains x and y).

The *closure* of a pair (x,y) of distinct points is defined as the set $\mathsf{cl}(x,y)$ of points which are collinear with a point in $\{x,y\}^{\perp\perp}$ (see [12, pp. 2]). In other words, for x, y non-collinear, it consists of the points on the lines between $\{x,y\}^\perp$ and $\{x,y\}^{\perp\perp}$. The following theorem will be very useful in Section 8 when we see how it applies to m-ovoids of certain generalised quadrangles.

Theorem 4.1 *Let x and y be two non-collinear points of a generalised quadrangle $\mathcal{G}$ of order (s,t). Then $\alpha\mathcal{X}_{\{x,y\}^{\perp\perp}} + \beta\mathcal{X}_{\{x,y\}^\perp}$ is a weighted tight set if and only if $\alpha s = \beta t$, $|\{x,y\}^{\perp\perp}| = s^2/t + 1$, and for z not in $\mathsf{cl}(x,y)$, $|\{x,y,z\}^\perp| = t/s + 1$. Moreover, in that case, it is a weighted $(\alpha+\beta)$-tight set.*

Proof By Lemma 2.1, we know that $\mathcal{X}_S := \alpha\mathcal{X}_{\{x,y\}^{\perp\perp}} + \beta\mathcal{X}_{\{x,y\}^\perp}$ is in $(V^-)^\perp$ (that is, S is a weighted tight set) if and only if $A\mathcal{X}_S = (s-1)\mathcal{X}_S + b\mathbf{j}$ for some $b \in \mathbb{C}$. We can rewrite this as

$$A\mathcal{X}_S = (\alpha(s-1)+b)\mathcal{X}_{\{x,y\}^{\perp\perp}} + (\beta(s-1)+b)\mathcal{X}_{\{x,y\}^\perp} + b(\mathbf{j} - \mathcal{X}_{\{x,y\}^{\perp\perp}} - \mathcal{X}_{\{x,y\}^\perp}).$$

It is not difficult to calculate geometrically that:

$\mathcal{X}_{p^\sim} \cdot \mathcal{X}_S = (t+1)\beta$, for $p \in \{x,y\}^{\perp\perp}$,
$\mathcal{X}_{p^\sim} \cdot \mathcal{X}_S = |\{x,y\}^{\perp\perp}|\alpha$, for $p \in \{x,y\}^\perp$,

$\mathcal{X}_{p^\sim} \cdot \mathcal{X}_S = \alpha + \beta$, for p on a line between $\{x,y\}^\perp$ and $\{x,y\}^{\perp\perp}$ but not in one of those two sets,
$\mathcal{X}_{p^\sim} \cdot \mathcal{X}_S = |\{x,y,p\}^\perp|\beta$, for the remaining points p (that is $p \notin \mathsf{cl}(x,y)$). Notice that such points are not collinear with any point in $\{x,y\}^{\perp\perp}$.

Assume first that S is a weighted tight set. Then we must have $\alpha(s-1)+b=(t+1)\beta$, $\beta(s-1)+b=|\{x,y\}^{\perp\perp}|\alpha$ and $b=\alpha+\beta=|\{x,y,p\}^\perp|\beta$. It follows easily that $\alpha s=\beta t$, that $|\{x,y\}^{\perp\perp}|=1+s^2/t$, and that $|\{x,y,z\}^\perp|=1+t/s$ for all $z\notin \mathsf{cl}(x,y)$.

Now assume $\alpha s=\beta t$, $|\{x,y\}^{\perp\perp}|=s^2/t+1$, and for $z\notin\mathsf{cl}(x,y)$, $|\{x,y,z\}^\perp|=t/s+1$. Then

$$\mathcal{X}_{p^\sim}\cdot\mathcal{X}_S=\begin{cases}(t+1)\beta=\alpha(s-1)+(\alpha+\beta) & \text{whenever } p\in\{x,y\}^{\perp\perp},\\ |\{x,y\}^{\perp\perp}|\alpha=\beta(s-1)+(\alpha+\beta) & \text{whenever } p\in\{x,y\}^{\perp},\\ \alpha+\beta & \text{whenever } p\in\mathsf{cl}(x,y)\backslash(\{x,y\}^{\perp\perp}\cup\{x,y\}^{\perp},\\ |\{x,y,p\}^\perp|\beta=\alpha+\beta & \text{whenever } p\notin\mathsf{cl}(x,y).\end{cases}$$

Therefore $A\mathcal{X}_S=(s-1)\mathcal{X}_S+(\alpha+\beta)\mathbf{j}$, and S is a weighted $(\alpha+\beta)$-tight set. □

By [12, §5.6.1], a generalised quadrangle of order (s,t) (with $s\neq 1$) satisfying $|\{x,y\}^{\perp\perp}|\geqslant s^2/t+1$ for all non-collinear pairs x and y must satisfy one of three cases: (i) have $t=s^2$, (ii) be isomorphic to $\mathsf{W}(3,q)$, or (iii) be isomorphic to $\mathsf{H}(4,q^2)$. Below we give some new proofs of two previously known results, and we provide one new result which we will use in Section 9.

Corollary 4.1 ([2, Lemma A.1]) *Let x and y be two non-collinear points of a generalised quadrangle of order (q,q^2), where $q>1$. Then*

$$q\mathcal{X}_{\{x,y\}}+\mathcal{X}_{\{x,y\}^\perp}$$

is a weighted tight set.

Proof By Bose and Shrikhande [4], the size of $\{x,y,z\}^\perp$ is $q+1$, where x,y,z are pairwise non-collinear. If there was a point z in $\{x,y\}^{\perp\perp}\setminus\{x,y\}$, then that point would be non-collinear to x and y and such that $|\{x,y,z\}^\perp|=q^2+1\neq q+1$. Thus $\mathcal{X}_{\{x,y\}^{\perp\perp}}=\mathcal{X}_{\{x,y\}}$. The result follows by Theorem 4.1. □

The following corollary also follows from [11, II.4]. It applies in particular to $\mathsf{W}(3,q)$, since for all pairs $\{x,y\}$ of non-collinear points $|\{x,y\}^{\perp\perp}|=q+1$ (c.f., [12, 3.3.1(i)]).

Corollary 4.2 *Let x and y be two non-collinear points of a generalised quadrangle of order (s,s), such that $|\mathcal{X}_{\{x,y\}^{\perp\perp}}|=s+1$. Then*

$$\mathcal{X}_{\{x,y\}^{\perp\perp}}+\mathcal{X}_{\{x,y\}^\perp}$$

is a 2-tight set.

Proof By simply counting the number of points on the lines between $\{x,y\}^\perp$ and $\{x,y\}^{\perp\perp}$, we see that

$$\begin{aligned}\mathsf{cl}(x,y)&=|\{x,y\}^\perp|+|\{x,y\}^{\perp\perp}|+\sum_{u\in\{x,y\}^{\perp\perp}}|\{z: z\in u^\sim\backslash\{x,y\}^\perp\}|\\&=2(s+1)+(s+1)(s+1)(s-1)\\&=(s+1)(s^2+1)\end{aligned}$$

and so $\mathsf{cl}(x,y)$ is the entire point-set. Therefore, the condition on $|\{x,y,z\}^\perp|$ for a point z not in $\mathsf{cl}(x,y)$ is vacuous and the result follows by Theorem 4.1. □

Corollary 4.3 *Let x and y be two non-collinear points of $\mathsf{H}(4,q^2)$. Then*

$$q\mathcal{X}_{\{x,y\}^{\perp\perp}}+\mathcal{X}_{\{x,y\}^\perp}$$

is a weighted tight set.

Proof Let z be a point not on a line between $\{x,y\}^{\perp\perp}$ and $\{x,y\}^\perp$. This condition on z ensures that the plane spanned by $\{x,y,z\}$ is non-degenerate, since otherwise, the line $(x\cap y)z$ is singular and the non-degenerate line containing x and y meets $(x\cap y)z$ in a singular point. So $\langle x,y,z\rangle^\perp$ is a non-degenerate line, and hence $|\{x,y,z\}^\perp|=q+1$. Note also that every hyperbolic line of $\mathsf{H}(4,q^2)$ has size $q+1$, which is $s^2/t+1$ for this generalised quadrangle. Finally, $t=qs$ and so the result follows by Theorem 4.1. □

5 Some characterisations of the direct summands of the points module

In this section, we give characterisations of the direct summands of the module on the points of a generalised quadrangle. As before, consider a generalised quadrangle $\mathcal{G}$ and let $\mathcal{P}$ and $\mathcal{L}$ be its sets of points and lines respectively. Recall that $\mathbb{C}\mathcal{P}$ decomposes into three submodules:

$$\mathbb{C}\mathcal{P} = \langle \mathbf{j} \rangle \oplus V^+ \oplus V^-.$$

Lemma 5.1 $V^+ = \langle \mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2} : \ell_1, \ell_2 \in \mathcal{L}, \ell_1 \cap \ell_2 \neq \varnothing \rangle$.

Proof Let $X = \langle \mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2} : \ell_1 \cap \ell_2 \neq \varnothing \rangle$. We have seen in Section 4.1 that for a line ℓ, $A\mathcal{X}_\ell = (s-1)\mathcal{X}_\ell + \mathbf{j}$. If ℓ_1 and ℓ_2 are any two lines, then $A(\mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2}) = (s-1)(\mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2})$. Hence

$$X \subseteq \langle \mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2} : \ell_1, \ell_2 \in \mathcal{L} \rangle \subseteq V^+.$$

Now let $f \in X^\perp$. Then $f \cdot \mathcal{X}_{\ell_1} = f \cdot \mathcal{X}_{\ell_2}$ for any two intersecting lines ℓ_1 and ℓ_2. Since the line graph of $\mathcal{G}$ is connected (it has diameter 2), it follows that $f \cdot \mathcal{X}_\ell = m$ is independent of the choice of ℓ. Then $f \in (V^+)^\perp$ by Lemma 4.2. Hence $X^\perp \subseteq (V^+)^\perp$, and so $V^+ \subseteq X$. This concludes the proof. □

Remark 5.1 Note that we have also proved that $V^+ = \langle \mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2} : \ell_1, \ell_2 \in \mathcal{L} \rangle$. Also for a fixed line $\ell_0 \in \mathcal{L}$, $V^+ = \langle \mathcal{X}_\ell - \mathcal{X}_{\ell_0} : \ell \in \mathcal{L} \rangle$. This follows from the fact that $\mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2} = (\mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_0}) - (\mathcal{X}_{\ell_2} - \mathcal{X}_{\ell_0})$ for any $\ell_1, \ell_2 \in \mathcal{L}$.

Lemma 5.2 $(V^-)^\perp = \langle \mathcal{X}_\ell \rangle$.

Proof We have seen in Section 4.1 that $\langle \mathcal{X}_\ell \rangle \subseteq (V^-)^\perp$. We will now show that $\langle \mathcal{X}_\ell \rangle^\perp \subseteq (V^-)$. Suppose $f \in \langle \mathcal{X}_\ell \rangle^\perp$. Then $\mathcal{X}_\ell \cdot f = 0$ for any line ℓ. By Lemma 4.2, $Af = -(t+1)f$ and thus $f \in V^-$. It follows that $\langle \mathcal{X}_\ell \rangle^\perp \subseteq V^-$, and so $(V^-)^\perp \subseteq \langle \mathcal{X}_\ell \rangle$. □

Remark 5.2 We could also prove Lemma 5.2 by using the incidence map $\iota : \mathbb{C}\mathcal{P} \to \mathbb{C}\mathcal{L}$. That is, for every $p \in \mathcal{P}$, we have $(\mathcal{X}_p)\iota = \sum_{\ell \sim p} \mathcal{X}_\ell$ and we extend ι linearly to $\mathbb{C}\mathcal{P}$. Let ι^* be the adjoint map of ι and let v be an eigenvector of A with eigenvalue $s-1$. Now $A = \iota\iota^* - (t+1)I$ and so $(v\iota)\iota^* - (t+1)v = (s-1)v$. This implies that $(s+t)v = (v\iota)\iota^*$ and hence v is in the image of ι^*. So $V^+ \subseteq \operatorname{Im}\iota^*$ and therefore $(V^-)^\perp = \langle \mathbf{j} \rangle + V^+ \subseteq \operatorname{Im}\iota^*$ (notice that $\mathbf{j} = (\frac{1}{t+1}\sum_{\ell \in \mathcal{L}} \mathcal{X}_\ell)\iota^*$ and hence $\mathbf{j} \in \operatorname{Im}\iota^*$).

For what follows, recall that C_x denotes the weighted cone seen in §4.5, that is, $\mathcal{X}_{C_x} = (1-s)\mathcal{X}_x + \mathcal{X}_{x^\sim}$.

Lemma 5.3 $V^- = \langle \mathcal{X}_{C_{x_1}} - \mathcal{X}_{C_{x_2}} : x_1, x_2 \in \mathcal{P} \rangle$.

Proof Let $U = \langle \mathcal{X}_{C_{x_1}} - \mathcal{X}_{C_{x_2}} : x_1, x_2 \in \mathcal{P} \rangle$. We have seen that for a cone C_x, $A\mathcal{X}_{C_x} = -(t+1)\mathcal{X}_{C_x} + (t+1)\mathbf{j}$. If C_{x_1} and C_{x_2} are any two cones, then $A(\mathcal{X}_{C_{x_1}} - \mathcal{X}_{C_{x_2}}) = -(t+1)(\mathcal{X}_{C_{x_1}} - \mathcal{X}_{C_{x_2}})$. Hence $U \subseteq V^-$.

Suppose $f \in U^\perp$. Then there is a constant b such that $\mathcal{X}_{C_x} \cdot f = b$ for any cone C_x. By Lemma 4.3, it follows that $f \in (V^-)^\perp$. Hence $U^\perp \subseteq (V^-)^\perp$, and so $V^- \subseteq U$. This concludes the proof. □

Remark 5.3 Let C_0 be a fixed cone. Then we also have $V^- = \langle \mathcal{X}_{C_x} - \mathcal{X}_{C_0} : x \in \mathcal{P} \rangle$.

Lemma 5.4 $(V^+)^\perp = \langle \mathcal{X}_{C_x} : x \in \mathcal{P} \rangle$.

Proof Let $K = \langle \mathcal{X}_{C_x} : x \in \mathcal{P} \rangle$. We have seen in Section 4.5 that any cone C_x is a weighted 1-ovoid, and so is in $(V^+)^\perp$. Thus $K \subseteq (V^+)^\perp$. Suppose $f \in K^\perp$. Then $\mathcal{X}_{C_x} \cdot f = 0$ for any cone C_x. By Lemma 4.3, $Af = (s-1)f$. Thus $f \in V^+$. It follows that $K^\perp \subseteq V^+$, and so $(V^+)^\perp \subseteq K$. □

6 A generalised quadrangle minus a cone

From the perspective of partial quadrangles, the following problem arose in [1]. Suppose $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ is a generalised quadrangle of order (s, s^2) and let p be a point of $\mathcal{G}$. If we consider the set $\mathcal{P}^p$ of points not collinear with p and the set of lines not incident with p, we produce a *partial quadrangle* $PQ(\mathcal{G})$, i.e., a point-line geometry such that for every point/line non-incident pair (p, ℓ), there is at most one line through p concurrent with ℓ. In particular, the point-graph of this object is strongly regular, so we can play the same game and define intriguing sets for this situation. The point-graph of $PQ(\mathcal{G})$ has eigenvalues and multiplicities listed below (see [1]):

Eigenvalue	Multiplicity
$(s-1)(s^2+1)$	1
$s-1$	$s(s-1)(s^2+1)$
$-s^2+s-1$	$(s-1)(s^2+1)$

Table 2 Eigenvalues for the point graph of the partial quadrangle $PQ(\mathcal{G})$.

We set $\mathbf{j}^p \in \mathbb{C}\mathcal{P}^p$ as the constant map with value 1. For a set S of points we denote by S^p the subset of S consisiting of points not collinear with p. By considering the adjacency matrices of both geometries, we have two direct decompositions

$$\mathbb{C}\mathcal{P} = \langle \mathbf{j} \rangle \oplus V^+ \oplus V^- \quad \text{and} \quad \mathbb{C}\mathcal{P}^p = \langle \mathbf{j}^p \rangle \oplus W^+ \oplus W^-$$

in accordance with their eigenvalues.

Now $\mathbb{C}\mathcal{P}^p$ embeds naturally into $\mathbb{C}\mathcal{P}$, however, it is not guaranteed that the eigenspaces W^+ or W^- will correspond in a natural way with V^+ or V^-. It is known that for a quotient of an equitable partition of $\mathcal{P}$ the eigenspaces for the quotient are the images of the eigenspaces for $\mathbb{C}\mathcal{P}$ under the quotient map (see [9, §9.5]). For derived structures such as the partial quadrangle arising from a generalised quadrangle, we do not have such strong information for how the eigenspaces correspond, however, the parameters of our generalised quadrangle are such that there is a partial correspondence between the two systems of eigenspaces. Let $R : \mathbb{C}\mathcal{P} \to \mathbb{C}\mathcal{P}^p$ be the restriction map (which is linear), and let R^* be its adjoint map, that is, the inclusion map from $\mathbb{C}\mathcal{P}^p$ to $\mathbb{C}\mathcal{P}$. In what follows, the actions of these operators on their associated function spaces will always be assumed to be on the left of elements. Note that R is surjective, R^* is injective, and RR^* is the identity map on $\mathbb{C}\mathcal{P}^p$. Moreover R^*R acts as the identity on the submodule $\mathbb{C}\mathcal{P}^p$ of $\mathbb{C}\mathcal{P}$, and zero on $\mathbb{C}p^\perp$.

For the rest of the section, we examine improvements of results of Section 5 of [1]. For instance, we have generalised [1, Lemma 5.6] by showing that $R(V^-) = \langle \mathbf{j}^p \rangle \oplus W^-$.

Lemma 6.1 *$R(V^-) = (W^+)^\perp$ and $R((V^+)^\perp) = (W^+)^\perp$, with $(\ker R) \cap V^- = \{0\}$ and $(\ker R) \cap (V^+)^\perp = \langle \mathcal{X}_{C_p} \rangle$.*

Proof We will think of R as the characteristic matrix for $\mathcal{P}^p$ (within $\mathcal{P}$). Then the adjacency matrix B of the partial quadrangle $PQ(\mathcal{G})$ is just RAR^T. Suppose $f \in V^-$, so $Af = (-s^2-1)f$. Let $f_{p^\perp}$ be the map

$$f_{p^\perp} : x \mapsto \begin{cases} f(x) & \text{if } x \in p^\perp \\ 0 & \text{otherwise,} \end{cases}$$

and let $f^p = Rf$, that is, the restriction of f to $\mathcal{P}^p$. We claim that $f^p \in (W^+)^\perp$.

By Corollary 4.1, $s\mathcal{X}_{\{x,p\}} + \mathcal{X}_{\{x,p\}^\perp} \in (V^-)^\perp$ for all $x \in \mathcal{P}^p$. Thus, for all $x \in \mathcal{P}^p$,

$$(s\mathcal{X}_{\{x,p\}} + \mathcal{X}_{\{x,p\}^\perp}) \cdot f = 0$$

and therefore

$$\mathcal{X}_{\{x,p\}^\perp} \cdot f = -s\mathcal{X}_x \cdot f - s\mathcal{X}_p \cdot f.$$

So for all $x \in \mathcal{P}^p$, we have $Af_{p^\perp} \cdot \mathcal{X}_x = \mathcal{X}_{x^\sim} \cdot f_{p^\perp} = \mathcal{X}_{x^\perp} \cdot f_{p^\perp} = -s\mathcal{X}_x \cdot f - s\mathcal{X}_p \cdot f$. When we apply R to the left of $Af_{p^\perp}$, we only have the values of it on $\mathcal{P}^p$:

$$RAf_{p^\perp} = \sum_{x \in \mathcal{P}^p} (-s\mathcal{X}_x \cdot f - s\mathcal{X}_p \cdot f)\mathcal{X}_x = -sf^p - s(\mathcal{X}_p \cdot f)\mathbf{j}^p.$$

So

$$\begin{aligned} Bf^p &= RA(R^TR)f = RA(f - f_{p^\perp}) \\ &= -(s^2+1)Rf - RAf_{p^\perp} \\ &= -(s^2+1)f^p + sf^p + s(\mathcal{X}_p \cdot f)\mathbf{j}^p \\ &= -(s^2-s+1)f^p + s(\mathcal{X}_p \cdot f)\mathbf{j}^p. \end{aligned}$$

Since $-(s^2-s+1)$ is the negative eigenvalue of B, by Lemma 2.1, $f^p \in (W^+)^\perp$. Therefore, $R(V^-) \subseteq (W^+)^\perp$. We will show that the dimensions of these spaces are equal.

Now $\ker R$ is spanned by the $\mathcal{X}_z$, where $z \in p^\perp$. Moreover, the $\mathcal{X}_z$ form a basis for $\ker R$ as $\dim \ker R = s(s^2+1)+1 = |p^\perp|$. Consider an arbitrary element $h := \sum_{z\in p^\perp} \alpha_z \mathcal{X}_z$ of $\ker R$, and suppose that $h \in V^-$. Then $Ah = \sum_{z\in p^\perp} \alpha_z \mathcal{X}_{z^\sim}$ and h is annihilated by $A + (s^2+1)I$:

$$\sum_{z\in p^\perp} \alpha_z \left(\mathcal{X}_{z^\sim} + (s^2+1)\mathcal{X}_z\right) = 0.$$

We claim that the $\mathcal{X}_{z^\sim} + (s^2+1)\mathcal{X}_z$ are linearly independent. Think of the square matrix M of size $|p^\perp|$, where rows correspond to the values of $\mathcal{X}_{z^\sim} + (s^2+1)\mathcal{X}_z$ restricted to $p^\perp$. So M is equal to the sum of $(s^2+1)I$ and the adjacency matrix D of the point graph of $\mathcal{G}$ restricted to $p^\perp$. Notice that this point graph can be seen as a single vertex (p) joined to all vertices of a graph isomorphic to $(s^2+1)K_s$ ($p^\perp \setminus \{p\}$). This is called a *complete product* in Cvetković, Doob and Sachs [6]. Theorem 2.8 of [6] gives us the characteristic polynomial $P_{\Gamma_1 \triangledown \Gamma_2}(\lambda)$ of a complete product $\Gamma_1 \triangledown \Gamma_2$ of two regular graphs. It is very easy to substitute K_1 into the formula to make it even simpler:

$$P_{\Gamma\triangledown K_1}(\lambda) = \frac{P_\Gamma(\lambda)}{\lambda - k}(\lambda(\lambda-k)-n)$$

where Γ is a regular graph of degree k and order n. Now we take $\Gamma = (s^2+1)K_s$. By Theorem 2.4 of [6], we have

$$P_\Gamma(\lambda) = P_{K_s}(\lambda)^{s^2+1} = \left((\lambda-s+1)(\lambda+1)^{s-1}\right)^{s^2+1}.$$

So putting it all together, we get the characteristic polynomial of D:

$$\begin{aligned} P_{\Gamma\triangledown K_1}(\lambda) &= \frac{(\lambda-s+1)^{s^2+1}(\lambda+1)^{(s-1)(s^2+1)}}{\lambda-s+1}(\lambda(\lambda-s+1)-s(s^2+1)) \\ &= (\lambda-s+1)^{s^2}(\lambda+1)^{(s-1)(s^2+1)}(\lambda^2+(1-s)\lambda-s(s^2+1)) \end{aligned}$$

So D has full rank and in particular $-s^2-1$ is not an eigenvalue of D. It follows that $M = D+(s^2+1)I$ has full rank. So the $\mathcal{X}_{z^\sim} + (s^2+1)\mathcal{X}_z$ are independent, as claimed. Therefore $h=0$ and $(\ker R)\cap V^- = \{0\}$.

Now $\dim V^- = s(s^2-s+1)$ and $\dim(W^+)^\perp = 1+(s-1)(s^2+1) = s(s^2-s+1)$. Therefore, $R(V^-) = (W^+)^\perp$.

Recall that $(V^+)^\perp = \langle \mathbf{j}\rangle \oplus V^-$. We also have that $R\mathbf{j} = \mathbf{j}^p \in (W^+)^\perp$, and so $R((V^+)^\perp) = (W^+)^\perp$ too. Obviously $\ker R \cap (V^+)^\perp$ is one-dimensional and $R\mathcal{X}_{C_p} = 0$. Since $\mathcal{X}_{C_p} \in (V^+)^\perp$ by §4.5, we have $\ker R \cap (V^+)^\perp = \langle \mathcal{X}_{C_p}\rangle$. □

The following theorem is a significant improvement on [1, Theorem 5.10], which we will explain in Remark 6.1. Recall that $\ell^p = \ell \cap \mathcal{P}^p$ and similarly $C_z^p = C_z \cap \mathcal{P}^p$.

Theorem 6.1 *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2) and let $PQ(\mathcal{G})$ be the related partial quadrangle with point set $\mathcal{P}^p = \mathcal{P}\backslash p^\perp$. Suppose we have a function $\overline{f} \in \mathbb{C}\mathcal{P}^p$ such that*

$$\overline{f} \in \langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle^\perp.$$

Then for any $m \in \mathbb{C}$, there exists an element f of $(V^+)^\perp$ such that, for any line ℓ, we have $\mathcal{X}_\ell \cdot f = m$ and

$$\overline{f} = Rf = f|_{\mathcal{P}^p}.$$

Proof Let B be the adjacency matrix of $PQ(\mathcal{G})$. Since $f|_{\mathcal{P}^p}$ is already determined, we only need to consider the values of $f|_{p^\perp}$. For $z \in p^\perp\setminus\{p\}$, we put $f\cdot\mathcal{X}_z = m - \frac{1}{s^2}\overline{f}\cdot\mathcal{X}_{C_z^p}$. We also put $f\cdot\mathcal{X}_p = m(1-s)+\frac{1}{s^2}\overline{f}\cdot\mathcal{X}_{\mathcal{P}^p}$.

We need to prove that for every $\ell \in \mathcal{L}$, $f \cdot \mathcal{X}_\ell = m$. Notice that all lines of $\mathcal{G}$ are either in $p^\perp$ or intersect $p^\perp$ in a single point, since $\mathcal{G}$ is a generalised quadrangle. Let ℓ_1, ℓ_2 be two lines of $\mathcal{G}$ intersecting $p^\perp$ in z. Notice that z cannot be p. By hypothesis, $\overline{f}\cdot(\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p}) = 0$, or in other words $\overline{f}\cdot\mathcal{X}_{\ell_1^p} = \overline{f}\cdot\mathcal{X}_{\ell_2^p}$. Since ${C_z}^p$ is the disjoint union of $t = s^2$ (intersections with $\mathcal{P}^p$ of) lines containing z, we have that $\overline{f}\cdot\mathcal{X}_{C_z^p} = s^2\overline{f}\cdot\mathcal{X}_{\ell_1^p}$. Thus $f\cdot\mathcal{X}_{\ell_1} = f\cdot\mathcal{X}_{\ell_1^p} + f\cdot\mathcal{X}_z = \frac{1}{s^2}\overline{f}\cdot\mathcal{X}_{C_z^p} + (m - \frac{1}{s^2}\overline{f}\cdot\mathcal{X}_{C_z^p}) = m$.

Now let ℓ be a line of $\mathcal{G}$ contained in $p^\perp$. Then let $\ell = \{p, z_1, z_2, \dots z_s\}$. We get

$$\begin{aligned} f \cdot \mathcal{X}_\ell &= \left(m(1-s) + \frac{1}{s^2}\overline{f} \cdot \mathcal{X}_{\mathcal{P}^p}\right) + \sum_{i=1}^{s}\left(m - \frac{1}{s^2}\overline{f} \cdot \mathcal{X}_{C^p_{z_i}}\right) \\ &= m(1-s) + \frac{1}{s^2}\overline{f} \cdot \mathcal{X}_{\mathcal{P}^p} + ms - \frac{1}{s^2}\sum_{i=1}^{s}\overline{f} \cdot \mathcal{X}_{C^p_{z_i}} \\ &= m + \frac{1}{s^2}\overline{f} \cdot (\mathcal{X}_{\mathcal{P}^p} - \sum_{i=1}^{s}\mathcal{X}_{C^p_{z_i}}). \end{aligned}$$

Note that every point in $\mathcal{P}^p$ is collinear with exactly one point of ℓ (and that point cannot be p), that is, the s "cones" $C^p_{z_i}$ partition $\mathcal{P}^p$. Hence $\sum_{i=1}^{s}\mathcal{X}_{C^p_{z_i}} = \mathcal{X}_{\mathcal{P}^p}$, and so $f \cdot \mathcal{X}_\ell = m$. By Lemma 4.2, $f \in (V^+)^\perp$, and this concludes the proof. □

Corollary 6.1 $W^+ = \langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle$.

Proof Let ℓ_1, ℓ_2 be two lines of $\mathcal{G}$ intersecting in $z \in p^\perp$. Then it is easy to see that $B\mathcal{X}_{\ell_1^p} = (s-1)\mathcal{X}_{\ell_1^p} + 1.(\mathbf{j}^p - \mathcal{X}_{C^p_z})$. Similarly $B\mathcal{X}_{\ell_2^p} = (s-1)\mathcal{X}_{\ell_2^p} + 1.(\mathbf{j}^p - \mathcal{X}_{C^p_z})$, and so $B(\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p}) = (s-1)(\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p})$. As $s - 1 > 0$, it means that $\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} \in W^+$. Hence $\langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle \subseteq W^+$.

Suppose $\overline{f} \in \mathbb{C}\mathcal{P}^p$ such that $\overline{f} \in \langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle^\perp$ and choose a constant m. Then by Theorem 6.1, $\overline{f}$ is the restriction to $\mathcal{P}^p$ of a function f in $(V^+)^\perp = \langle \mathbf{j} \rangle \oplus V^-$ such that, for any line ℓ, we have $\mathcal{X}_\ell \cdot f = m$. By Lemma 6.1, $\overline{f} = Rf \in (W^+)^\perp$. Thus $\langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle^\perp \subseteq (W^+)^\perp$, and so $W^+ \subseteq \langle \mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p} : \ell_1 \cap \ell_2 \in p^\perp \rangle$. This concludes the proof. □

Corollary 6.2 $R^*(W^+) \subseteq V^+$.

Proof By Corollary 6.1, it is enough to prove that $R^T(\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p}) \subseteq V^+$ for all lines ℓ_1, ℓ_2 such that $\ell_1 \cap \ell_2 \in p^\perp$. It is easy to see that $R^T(\mathcal{X}_{\ell_1^p} - \mathcal{X}_{\ell_2^p}) = \mathcal{X}_{\ell_1} - \mathcal{X}_{\ell_2}$ and so the result follows directly by Lemma 5.1. □

Corollary 6.3 *Suppose we have a weighted set $\overline{\mathcal{S}}$ of $\mathcal{P}^p$ such that $\mathcal{X}_{\overline{\mathcal{S}}} \in (W^+)^\perp$. Then for any $m \in \mathbb{Z}$, there exists a weighted m-ovoid $\mathcal{S}$ of $\mathcal{G}$ such that*

$$\mathcal{X}_{\overline{\mathcal{S}}} = (\mathcal{X}_{\mathcal{S}})|_{\mathcal{P}^p}.$$

Proof In Theorem 6.1, we constructed a function f satisfying $\mathcal{X}_{\overline{\mathcal{S}}} = f|_{\mathcal{P}^p}$ and, for any line ℓ, $\mathcal{X}_\ell \cdot f = m$. We only have to show that f is the characteristic function of a weighted set.

Since $f|_{\mathcal{P}^p} = \mathcal{X}_{\overline{\mathcal{S}}}$, f has integral values on $\mathcal{P}^p$. For $z \in p^\perp \setminus \{p\}$, $f \cdot \mathcal{X}_z = m - \frac{1}{s^2}\mathcal{X}_{\overline{\mathcal{S}}} \cdot \mathcal{X}_{C^p_z}$. Looking at the proof of Theorem 6.1, we see that $f \cdot \mathcal{X}_z = m - \mathcal{X}_{\overline{\mathcal{S}}} \cdot \ell_1$, where ℓ_1 is one of the lines through z not in $p^\perp$, thus this is an integer. Finally $f \cdot \mathcal{X}_p = f \cdot \mathcal{X}_\ell - \sum_{i=1}^{s} f \cdot \mathcal{X}_{z_i}$, where $\ell = \{p, z_1, z_2, \dots z_s\}$ is a line through p. Since each component of that sum is an integer, so is $f \cdot \mathcal{X}_p$. This concludes the proof. □

Remark 6.1 To summarise Lemma 6.1, Theorem 6.1 and Corollary 6.3, we have the following generalisation and simpler proof of [1, Theorem 5.10].

Corollary 6.4 *Let $\mathcal{G}$ be a generalised quadrangle of order (s, s^2). Let $\mathcal{P}$ be the point-set of $\mathcal{G}$ and let $\mathcal{P}^p$ be the point-set of the partial quadrangle $PQ(\mathcal{G})$.*

(i) If $f \in (V^+)^\perp$, then $Rf = f^p \in (W^+)^\perp$.
(ii) For any $m \in \mathbb{C}$, every element $\overline{f} \in (W^+)^\perp$ lifts to an element $f \in (V^+)^\perp$ such that for any line ℓ we have $\mathcal{X}_\ell \cdot f = m$. That is, $\overline{f} = f|_{\mathcal{P}^p}$. Moreover, suppose we have an unweighted set of points $\overline{\mathcal{S}}$ such that $\mathcal{X}_{\overline{\mathcal{S}}} \in (W^+)^\perp$. Then there exists a weighted m-ovoid $\mathcal{S}$ of $\mathcal{G}$ such that

$$\mathcal{X}_{\overline{\mathcal{S}}} = (\mathcal{X}_{\mathcal{S}})|_{\mathcal{P}^p}.$$

Remark 6.2 It was conjectured in [1, Conjecture 5.8] that if S is a weighted m-ovoid of $\mathcal{G}$ such that S^p (whose characteristic vector is in $(W^+)^\perp$ by Theorem 6.1) is not weighted, then S is a hemisystem or a union of cones. By computer, we have found a counter-example in the dual of the Fisher-Thas-Walker-Kantor-Betten generalised quadrangle of order $(5, 5^2)$. We do not know if a similar example exists in $\mathsf{Q}^-(5, q)$.

7 Payne-derived generalised quadrangles

Suppose we have two non-collinear points x and y of a generalised quadrangle $\mathcal{G}$. Recall that $|\{x,y\}^\perp| = t+1$, and $2 \leqslant |\{x,y\}^{\perp\perp}| \leqslant t+1$. We say a point x is *regular* if for every point y not collinear with x we have $|\{x,y\}^{\perp\perp}| = t+1$. For example, every point of $\mathsf{W}(3,q)$ is regular, and no point of $\mathsf{Q}(4,q)$ is regular for q odd. Given a generalised quadrangle $\mathcal{G}$ of order (s,s), and a regular point x of $\mathcal{G}$, we can construct the *Payne derived* generalised quadrangle $\mathcal{G}^x$ as follows (c.f., [12, §3.1.4]):

POINTS	$\mathcal{P}^x$, the points of $\mathcal{G}$ not in $x^\perp$
LINES	(i) The lines of $\mathcal{G}$ not incident with x (ii) The *hyperbolic lines* $\{x,y\}^{\perp\perp}$
INCIDENCE	Inherited from $\mathcal{G}$

Clearly $\mathcal{G}^x$ has order $(s-1,s+1)$, and thus the point graph has eigenvalues and multiplicities as follows:

Eigenvalue	Multiplicity
$(s-1)(s+2)$	1
$s-2$	$\frac{(s^2-1)(s+2)}{2}$
$-s-2$	$s\frac{(s-1)^2}{2}$

Recall that ovoids are 1-ovoids, that is, they are sets of points of $\mathcal{G}$ intersecting every line of $\mathcal{G}$ in a point.

Lemma 7.1 ([12, 3.4.3]) *Consider a generalised quadrangle $\mathcal{G}$ of order (s,s) and let x be a regular point. Let $\mathcal{O}$ be an ovoid of $\mathcal{G}$ containing x. Then $\mathcal{O}\backslash\{x\}$ is an ovoid of $\mathcal{G}^x$.*

Proof We have to show that all lines of $\mathcal{G}^x$ meet $\mathcal{O}\backslash\{x\}$ in exactly one point. First let ℓ be a line of $\mathcal{G}$ not incident with x. Then ℓ meets $\mathcal{O}$ in a unique point, and this point is not in $x^\perp$, so in $\mathcal{G}^x$, ℓ meets $\mathcal{O}\backslash\{x\}$ in a unique point.

Now consider a hyperbolic line $\{x,y\}^{\perp\perp}$, where y is some element of $\mathcal{G}^x$. By Corollary 4.2, $\{x,y\}^\perp \cup \{x,y\}^{\perp\perp}$ forms a 2-tight set. By [3, Theorem 4.3], this 2-tight set meets the 1-ovoid $\mathcal{O}$ in two points, one of which is x. Let z be the other one. If z was in $x^\perp$, then the line xz would meet $\mathcal{O}$ in two points, a contradiction. Hence z is not in $x^\perp$ (and so it is in $\mathcal{P}^x$). In particular, z is not in $\{x,y\}^\perp$ and so $z \in \{x,y\}^{\perp\perp}$. Thus the hyperbolic line $\{x,y\}^{\perp\perp}$ meets $\mathcal{O}\backslash\{x\}$ in a unique point. □

Note that this proof is very similar to the one in [12]. We give it here for sake of completeness and as an illustration of the usefulness of tight sets and m-ovoids.

Lemma 7.2 *Consider the generalised quadrangle $\mathcal{G} := \mathsf{W}(3,q)$, where q is even, and let x be a point of $\mathcal{G}$. Then $\mathcal{G}^x$ can be partitioned into ovoids.*

Proof We will first note that $\mathcal{G}$ is isomorphic to the generalised quadrangle $\mathsf{Q}(4,q)$ and that x is regular (since all points are). Suppose $\mathsf{Q}(4,q)$ is defined by the quadratic form

$$Q(v) = v_0^2 + v_1v_2 + v_3v_4$$

and let $x = [0,0,0,0,1]$. Let α be an element of $\mathsf{GF}(q)$ with absolute trace equal to 1. For each $\lambda \in \mathsf{GF}(q)$, consider the following hyperplane π_λ of $\mathsf{PG}(4,q)$:

$$\pi_\lambda : v_0 + \alpha(v_1+v_2) + \lambda v_3 = 0.$$

Note that x lies on each of these hyperplanes. A point y of $\mathsf{Q}(4,q)$ is not collinear with x if and only if it is of the form

$$y = [y_0, y_1, y_2, 1, y_4]$$

and hence y is incident with a unique π_λ (with $\lambda = y_0 + \alpha(y_1+y_2)$). Hence the sets π_λ partition the pointset of $\mathcal{G}^x$. We are now going to prove that $\mathcal{O}_\lambda := \pi_\lambda \cap \mathcal{P}$ is an ovoid. By Lemma 7.1, we will then have that $\pi_\lambda \cap \mathcal{P}^x$ is an ovoid of $\mathcal{G}^x$.

Suppose $\mathcal{O}_\lambda$ is not an ovoid. Every (totally isotropic) line of $\mathsf{Q}(4,q)$ intersects π_λ, so every line of $\mathcal{G}$ contains at least one point of $\mathcal{O}_\lambda$. Therefore there is one line ℓ of $\mathcal{G}$ containing at least two points of $\mathcal{O}_\lambda$. By symmetry we can assume ℓ to contain x. Let $y = [y_0, y_1, y_2, y_3, y_4]$ be a point of $\ell \setminus \{x\}$ in $\mathcal{O}_\lambda$. Since y is collinear with x but is not x, $y_3 = 0$ and $(y_0, y_1, y_2) \neq (0,0,0)$. Since y is a point of $\mathcal{G}$, $y_0^2 + y_1 y_2 = 0$, and since y in π_λ, $y_0 + \alpha(y_1 + y_2) = 0$. Obviously $y_1 \neq 0$ and $y_2 \neq 0$. Then $w = \alpha^2 y_2/y_1 \neq 0$ satisfies $w^2 + w + \alpha^4 = 0$. Taking the trace, we get $\mathsf{Tr}(w^2) + \mathsf{Tr}(w) + \mathsf{Tr}(\alpha^4) = 0$. Since the "square" map is a field automorphism, $\mathsf{Tr}(w^2) = \mathsf{Tr}(w)$ and $\mathsf{Tr}(\alpha^4) = \mathsf{Tr}(\alpha) = 1$, and so we have a contradiction. Therefore $\mathcal{O}_\lambda$ is an ovoid. □

Corollary 7.1 *Consider the generalised quadrangle $\mathcal{G} := \mathsf{W}(3,q)$, where q is even, and let x be a point of $\mathcal{G}$. Then $\mathcal{G}^x$ has m-ovoids for every possible value m.*

Proof It follows from Lemma 7.2 and the fact that the disjoint union of m ovoids is an m-ovoid. □

Remark 7.1 By [12, 3.4.3], $\mathcal{G}^x$ contains a spread, that is, $\mathcal{P}^x$ can be partitioned into lines, which are each 1-tight sets. Since the disjoint union of i lines is an i-tight set, it follows that $\mathcal{G}^x$ contains i-tight sets for any positive integer i.

8 m-ovoids of generalised quadrangles with restricted hyperbolic line size

This section was motivated by the forthcoming Section 9 on m-ovoids of $\mathsf{H}(4,q^2)$, however, we found that our techniques could be extended to any generalised quadrangle which had a restricted hyperbolic line size. In fact, a simple corollary of the following theorem is Theorem 1.1 (see Corollary 8.2). Recall that S^x denotes $S \backslash x^\perp$

Theorem 8.1 *Let S be an m-ovoid of a generalised quadrangle of order (s,t), let x be a point lying outside of S. Suppose that for every $y \in S\backslash x^\perp$, $|\{x,y\}^{\perp\perp}| = s^2/t + 1$ and $|\{x,y,u\}^\perp| = t/s + 1$ for any u not in the closure of x and y. Then*

$$\sum_{z \in S^x} |S \cap \{x,z\}^{\perp\perp}| = m^2(s^2 - 2s - t) + ms(t+1).$$

Proof Fix a point x outside of S, and let $c = s^2/t + 1$. We will use a double counting argument, but because we will be working with multisets, we will explicitly state the double counting argument as the two ways of calculating the sum of all elements of a matrix M. For each $y \in S\backslash x^\perp$, define v_y to be the vector

$$t\mathcal{X}_{\{x,y\}^{\perp\perp}} + s\mathcal{X}_{\{x,y\}^\perp}.$$

Recall that v_y is a weighted tight set by Theorem 4.1. These vectors will give us the rows of our matrix M, except we will restrict the columns of our matrix to S. There are $|S| = m(st+1)$ columns and $|S| - |S \cap x^\perp| = m(st+1) - m(t+1) = mt(s-1)$ rows.

Counting by rows: The sum of the elements of each row is a constant as for $y \in S^x$, each v_y is a weighted tight set and hence we apply Lemma 2.2:

$$v_y \cdot \mathcal{X}_S = \frac{(\mathcal{X}_S \cdot \mathbf{j})(v_y \cdot \mathbf{j})}{|\mathcal{P}|} = \frac{|S|(v_y \cdot \mathbf{j})}{(s+1)(st+1)} = \frac{m(st+1)(tc + s(t+1))}{(s+1)(st+1)} = \frac{m(s+t)(s+1)}{s+1} = m(s+t)$$

and the sum of the elements of M is $m^2 t(s-1)(s+t)$.

Counting by columns: In this case, there are two possible values for the sum of the elements of a column. Consider a point $z \in S$. Then the corresponding column sum is

$$N_z := \sum_{y \in S^x} v_y \cdot \mathcal{X}_z.$$

If $z \in x^\perp$, then

$$N_{z \in x^\perp} = \sum_{y \in S^x} s\mathcal{X}_{\{x,y\}^\perp} \cdot \mathcal{X}_z = s|(S^x) \cap z^\perp| = (m-1)st.$$

If $z \notin x^\perp$, then

$$N_{z \notin x^\perp} = \sum_{y \in S^x} t\mathcal{X}_{\{x,y\}^{\perp\perp}} \cdot \mathcal{X}_z = t \sum_{y \in S^x} \mathcal{X}_{\{x,z\}^{\perp\perp}} \cdot \mathcal{X}_y = t|S \cap \{x,z\}^{\perp\perp}|.$$

We used here the fact that the hyperbolic line spanned by x and y is the same as the hyperbolic line spanned by x and z.

So we have in total, that the sum of the elements of M is

$$|S \cap x^\perp| N_{z \in x^\perp} + \sum_{z \in S^x} N_{z \notin x^\perp} = m(t+1)(m-1)st + t \sum_{z \in S^x} |S \cap \{x,z\}^{\perp\perp}|.$$

Double count: Now putting our two calculations together, we see that

$$\sum_{z \in S^x} |S \cap \{x,z\}^{\perp\perp}| = m^2(s-1)(s+t) - m(m-1)s(t+1) = m^2(s^2-2s-t) + ms(t+1).$$

□

Corollary 8.1 *Let S be an m-ovoid of a generalised quadrangle and let x be a point outside of S. Then for the generalised quadrangles below, we have the following values for $\sum_{z \in S \setminus x^\perp} |S \cap \{x,z\}^{\perp\perp}|$:*

$\mathsf{W}(3,q)$	$GQ(s,s^2)$	$\mathsf{H}(4,q^2)$
$mq\,(m(q-3)+q+1)$	$ms(s^2-2m+1)$	$mq^2(q+1)\big(m(q-2)+q^2-q+1\big)$

Proof It follows from the following table of values, where y is not collinear with x and u is not in the closure of x and y. See §4.7 for details.

GQ	s	t	$\lvert\{x,y\}^{\perp\perp}\rvert$	$\lvert\{x,y,u\}^{\perp}\rvert$
$\mathsf{W}(3,q)$	q	q	$q+1$	N/A
$GQ(s,s^2)$	s	s^2	2	$s+1$
$\mathsf{H}(4,q^2)$	q^2	q^3	$q+1$	$q+1$

□

This allows us to reprove Theorem 1.1.

Corollary 8.2 *An (non-trivial) m-ovoid of a generalised quadrangle of order (s,s^2) is a hemisystem.*

Proof Let S be a non-trivial m-ovoid, that is $m \neq 0$ and $m \neq s+1$. Then S does not cover the whole point-set, and so we can fix $x \notin S$. Recall that for z non-collinear with x, we have $\{x,z\}^{\perp\perp} = \{x,z\}$. So

$$|S^x| = \sum_{z \in S^x} |S \cap \{x,z\}^{\perp\perp}| = ms(s^2-2m+1).$$

On the other hand, $|S^x| = mt(s-1) = ms^2(s-1)$. Hence $s^2-2m+1 = s(s-1)$. This equation reduces to $2m-(s+1)=0$, and so $m=(s+1)/2$. □

Remark 8.1 If we suppose in Theorem 8.1 that $t=s^2$ and $\{x,z\}^{\perp\perp} = \{x,z\}$ (for all $z \notin x^\perp$), the proof radically reduces to a simple direct proof of Corollary 8.2. By Corollary 4.1, $v_y = s\mathcal{X}_{\{x,y\}} + \mathcal{X}_{\{x,y\}^\perp}$ is a weighted $(s+1)$-tight set, and so

$$|\{x,y\}^\perp \cap S| = \mathcal{X}_{\{x,y\}^\perp} \cdot \mathcal{X}_S = (v_y - s\mathcal{X}_{\{x,y\}}) \cdot \mathcal{X}_S = m(s+1) - s\mathcal{X}_y \cdot \mathcal{X}_S.$$

Now we double count pairs (y,z) where $y,z \in S$, $y \notin x^\perp$, with the condition that z lies in $\{x,y\}^\perp$:

(i) $\sum_{y \in S^x} |S \cap \{x,y\}^\perp| = |S^x|(m(s+1)-s) = ms^2(s-1)(m(s+1)-s)$
(ii) $\sum_{z \in S \cap x^\perp} |(z^\perp \setminus \ell_{xz}) \cap S| = |S \cap x^\perp| \cdot s^2(m-1) = m(m-1)s^2(s^2+1)$, where ℓ_{xz} is the line containing x and z.

So $(s-1)(m(s+1)-s) = (m-1)(s^2+1)$, which reduces to $2m = s+1$.

This proof is similar (but proved independently) to that of Vanhove [15, Theorem 3], if you consider generalised quadrangles of order (s,s^2) as particular cases of regular near polygons.

9 m-ovoids of the 4-dimensional Hermitian variety

Here we use counting arguments to give a non-existence result on m-ovoids of $\mathsf{H}(4,q^2)$. We obtain essentially the same bound as the one obtained in [3, Proof of Theorem 7.1], whereby our lower bound is always larger but unfortunately does not exclude more integers. It might be possible to improve on this bound by adjusting our argument, however, so far no improvement has been found this way.

The quadrangle $\mathsf{H}(4,q^2)$ has order (q^2,q^3) and its hyperbolic lines have size $q+1$ (they are the non-degenerate lines relative to the Hermitian polarity defining the quadrangle). Moreover it can easily be computed that the number of hyperbolic lines through a point is q^6 and the total number of hyperbolic lines is $q^6(q^5+1)(q^2+1)/(q+1)$.

Theorem 9.1 *Let S be a non-trivial m-ovoid of $\mathsf{H}(4,q^2)$. If $q \neq 2$, then*

$$m \geqslant \frac{1}{2}\,\frac{-3\,q-3+\sqrt{4\,q^5-4\,q^4+5\,q^2-2\,q+1}}{q^2-q-2},$$

while for $q=2$ we have $m \geqslant 2$.

Proof Let I be an index set for the hyperbolic lines of $\mathsf{H}(4,q^2)$. For each hyperbolic line h_i, $i \in I$, define

$$y_i := |S \cap h_i|.$$

Recall that S^p denotes the set of points in S not collinear with a given point $p \in S$. By Lemma 4.2, $|S| = m(q^5+1)$ and $|S^p| = |S| - (q^3+1)(m-1) - 1 = m(q^5-q^3)+q^3$ (which does not depend on p).

Counting the pairs (h,p) where h is a hyperbolic line and $p \in S \cap h$, we obtain

$$\sum_{i\in I} y_i = q^6|S| = q^6 m(q^5+1).$$

Counting the triples (h,p_1,p_2) where h is a hyperbolic line and $p_1,p_2 \in S\cap h$ with $p_1 \neq p_2$, we obtain

$$\sum_{i\in I} y_i(y_i-1) = |S||S'| = m(q^5+1)q^3(m(q^2-1)+1),$$

which implies that

$$\sum_{i\in I} y_i^2 = m(q^5+1)q^3(m(q^2-1)+q^3+1).$$

By Corollary 8.1, for a given $x \notin S$, we have

$$\sum_{z\in S^x} |S\cap\{x,z\}^{\perp\perp}| = mq^2(q+1)(m(q-2)+q^2-q+1).$$

Therefore a double counting argument yields

$$\sum_i y_i^2(q+1-y_i) = \sum_{x\notin S}\sum_{z\in S\setminus x^\perp} |S\cap\{x,z\}^{\perp\perp}| = (q^5+1)(q^2+1-m)mq^2(q+1)(m(q-2)+q^2-q+1)$$

and hence

$$\begin{aligned}\sum_i y_i^3 &= (q+1)\sum_i y_i^2 - \sum_{x\notin S}\sum_{z\in S\setminus x^\perp} |S\cap\{x,z\}^{\perp\perp}| \\ &= mq^2(q+1)(q^5+1)((q-2)m^2+3(q^2-q+1)m+q^3-2q^2+2q-1).\end{aligned}$$

The fact that $\sum y_i(y_i-1)(y_i-2) = \sum_i y_i^3 - 3\sum_i y_i^2 + 2\sum_i y_i$ has to be positive yields

$$0 \leqslant (q-2)(q+1)m^2 + 3(q+1)m - q^3 - 2q - 1.$$

Hence, for $q \neq 2$, $m \geqslant \frac{1}{2}\,\frac{-3\,q-3+\sqrt{4\,q^5-4\,q^4+5\,q^2-2\,q+1}}{q^2-q-2}$, while for $q=2$ the condition yields $m \geqslant 13/9$, and so $m \geqslant 2$. □

10 m-ovoids of $\mathsf{DH}(4, q^2)$

One of the most prominent open problems in finite geometry is whether a spread exists of the generalised quadrangle $\mathsf{H}(4, q^2)$. Currently, we only know of a computer result by Brouwer (see [12, pp. 47]) that there is no spread of $\mathsf{H}(4, 2^2)$. We may also ask the more general question of the existence of m-ovoids in the dual generalised quadrangle $\mathsf{DH}(4, q^2)$; the case $m = 1$ gives us precisely the question on the existence of spreads of $\mathsf{H}(4, q^2)$.

By computer, we have found many m-ovoids of $\mathsf{DH}(4, q^2)$ which are stabilised by a Singer type element, and we will describe how this was done in what follows. Consider the field $F := \mathsf{GF}(q^{10})$ and let $\mathsf{Tr}_{q^{10}\to q^2}$ be the relative trace map from F to $\mathsf{GF}(q^2)$. Define the following map β from $F \times F \to \mathsf{GF}(q^2)$:

$$\beta(x, y) := \mathsf{Tr}_{q^{10}\to q^2}(xy^{q^5}).$$

Notice that F can be written as a five-dimensional vector space V over $\mathsf{GF}(q^2)$, and β induces a Hermitian form on V. This is the model we will use for $\mathsf{H}(4, q^2)$, so the points $\mathcal{P}$ are the totally isotropic 1-spaces and the lines $\mathcal{L}$ are the totally isotropic 2-spaces relative to that Hermitian form. Let ζ be a primitive root of F and let $\omega = \zeta^{(q^5-1)(q+1)}$. The element ω is known as a *Singer type element*, that is, $K := \langle\omega\rangle$ acts irreducibly on V. Notice the stabiliser K_ℓ of a line is contained in the stabiliser of a line in the group $\mathrm{P\Gamma L}(5, q^2)$, and so has order dividing $(q^2)^2 - 1$. Therefore, $|K_\ell|$ divides the greatest common divisor of $q^4 - 1$ and $(q^5 + 1)/(q + 1)$, which is trivial. Hence K acts semiregularly on lines of $\mathsf{H}(4, q^2)$ and the orbits of K each have size $(q^5 + 1)/(q + 1)$. Let $\mathcal{O}$ be the set of those orbits. We will look for m-covers of $\mathsf{H}(4, q^2)$ which are K-invariant. An *m-cover* is a set of lines such that every point is in exactly m lines of the set, so it is the dual notion of an m-ovoid.

Let A be the concurrency matrix[1] of $\mathsf{H}(4, q^2)$, so it is the adjacency matrix of $\mathsf{DH}(4, q^2)$. Let P be the matrix whose rows are indexed by the lines of $\mathsf{H}(4, q^2)$ and whose columns are indexed by $\mathcal{O}$, where $P_{ij} = 1$ if the i-th line lies in the j-th orbit, and 0 otherwise. That is, P is the *characteristic matrix* for the orbit partition induced by the action of K on lines. Now $\frac{q+1}{q^5+1}P^TAP$ is the *collapsed adjacency matrix* C for the K-quotient of the concurrency graph, that is $(C)_{ij}$ is the number of lines in the j-th orbit which are concurrent to a given line in the i-th orbit. By [8, Lemma 2.2] and since C is symmetric, C has the same eigenvalues as A.

Now suppose S is an m-cover of $\mathsf{H}(4, q^2)$ which is K-invariant. Then S is an m-ovoid of $\mathsf{DH}(4, q^2)$, and by Lemma 4.2,

$$A\mathcal{X}_S = -(q^2 + 1)\mathcal{X}_S + m(q^2 + 1)\mathcal{X}_\mathcal{L}.$$

Since S is K-invariant, it follows that $(PP^T)\mathcal{X}_S = \frac{q^5+1}{q+1}\mathcal{X}_S$. So

$$\begin{aligned} C(P^T\mathcal{X}_S) &= \left(\frac{q+1}{q^5+1}P^TA\right)(PP^T)\mathcal{X}_S \\ &= -(q^2 + 1)P^T\mathcal{X}_S + m(q^2 + 1)P^T\mathcal{X}_\mathcal{L} \\ &= -(q^2 + 1)P^T\mathcal{X}_S + m(q^2 + 1)\left(\frac{q^5 + 1}{q + 1}\right)\mathcal{X}_\mathcal{O}. \end{aligned}$$

Now $JP^T\mathcal{X}_S = m(q^5 + 1)\mathcal{X}_\mathcal{O}$, where J is the $|\mathcal{O}| \times |\mathcal{O}|$ "all ones" matrix, and so a simple calculation shows that $P^T\mathcal{X}_S \in \ker(N)$ where

$$N := (q + 1)C + (q^2 + 1)(q + 1)I - (q^2 + 1)J.$$

Thus $x = \frac{q+1}{q^5+1}P^T\mathcal{X}_S$ is a $0-1$-vector in $\ker(N)$, such that $|x| = x \cdot \mathcal{X}_\mathcal{O} = m(q + 1)$.

Therefore, to find an m-cover of $\mathsf{H}(4, q^2)$ amounts to solving an integer linear program:

$$\begin{aligned} \texttt{maximise:} \quad & b^Tx \\ \texttt{subject to:} \quad & Nx = 0, \quad |x| = m(q + 1). \end{aligned}$$

where x is a $\{0, 1\}$-vector. The first condition is superfluous so we take $b = 0$. There exists much integer linear programming software that are freely available, and we used *Gurobi Optimiser 4.0* [10] in our search for K-invariant m-covers of $\mathsf{H}(4, q^2)$. Moreover, we found the following interesting phenomenon:

[1] This is the symmetric matrix corresponding to the concurrency relation. The (i, j)-entry of A is equal to 1 if the i-th line meets the j-th line in just one point, and 0 otherwise.

Lemma 10.1 *For $q \in \{2,3,4\}$ there exist K-invariant m-covers of $\mathsf{H}(4,q^2)$ for all m satisfying $n_q < m < q^3+1-n_q$, where $n_2 = n_3 = 2$ and $n_4 = 4$. There exists a K-invariant 5-cover of $\mathsf{H}(4,5^2)$, but no smaller K-invariant m-cover.*

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