A note on quasi-Hermitian varieties and singular Quasi-quadrics

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Abstract

Quasi-quadrics were introduced by Penttila, De Clerck, O'Keefe and Hamilton in [2]. They are defined as point sets which have the same intersection numbers with respect to hyperplanes as non-singular quadrics. We extend this definition in two ways.

The first extension is to *quasi-Hermitian varieties*, which are point sets which have the same intersection numbers with respect to hyperplanes as non-singular Hermitian varieties.

The second one is to *singular quasi-quadrics*, i.e. point sets $\mathcal{K}$ which have the same intersection numbers with respect to hyperplanes as singular quadrics. Our starting point was to investigate whether every singular quasi-quadric is a cone over a non-singular quasi-quadric. This question is tackled in the case of a point set $\mathcal{K}$ with the same intersection numbers with respect to hyperplanes as a point over an ovoid.

1 Introduction

In [2] quasi-quadrics were introduced, i.e. point sets $\mathcal{K}$ in $\mathrm{PG}(n, q)$ which have the same intersection numbers with respect to hyperplanes as non-singular quadrics. In that paper there is a free construction of these structures, yielding an overwhelming amount of examples. In this paper we define quasi-Hermitian varieties, i.e. the analogous concept of quasi-quadrics for Hermitian varieties and provide similar free constructions of them.

In [4], we proved that if one additionally assumes that $\mathcal{K}$ has the same intersection numbers with respect to spaces of codimension 2 as non-singular quasi-quadrics (non-singular Hermitian varieties), then $\mathcal{K}$ is a non-singular quadric (non-singular Hermitian variety).

The goal of this paper is to extend the theory to singular quadrics (Hermitian varieties). We prove similar results in the low-rank case.

2 Quasi-Hermitian varieties

Definition 1
A set of points H in $PG(n,q^2)$, with size equal to the number of points lying on a non-singular Hermitian variety $H(n,q^2)$, is called a quasi-Hermitian variety in $PG(n,q^2)$, if its intersection numbers with hyperplanes are the size of a non-singular Hermitian variety $H(n-1,q^2)$, namely

$$\frac{(q^n+(-1)^{n-1})(q^{n-1}-(-1)^{n-1})}{q^2-1}$$

or the size of a cone with vertex a point p and base a non-singular Hermitian variety $H(n-2,q^2)$, shortly denoted by $pH(n-2,q^2)$, namely

$$\frac{(q^n+(-1)^{n-1})(q^{n-1}-(-1)^{n-1})}{q^2-1}+(-1)^{n-1}q^{n-1}$$

We will call hyperplanes intersecting H in $|H(n-1,q^2)|$ points *secant*, the other ones *tangent*.

We will show that not all quasi-Hermitian varieties are Hermitian varieties. Our first construction is the Hermitian analogue of a construction method by Penttila, De Clerck, O'Keefe and Hamilton, a method which they call pivotting.

Let $H(n,q^2)$ be a non-singular Hermitian variety. Take a point p on $H(n,q^2)$ and consider the tangent space Π of the Hermitian variety at p. This space intersects the Hermitian variety in a cone with vertex p and base a non-singular Hermitian variety $H(n-2,q^2)$ lying in a $PG(n-2,q^2)$. We replace this non-singular Hermitian variety $H(n-2,q^2)$ by a quasi-Hermitian variety in $PG(n-2,q^2)$, say H'. We call the set of points contained in $(H(n,q^2)-pH(n-2,q^2))\cup pH'$ a pivotted set of $H(n,q^2)$ with respect to p.

Theorem 2
Every pivotted set of $H(n,q^2)$ with respect to a point p of $H(n,q^2)$ is a quasi-Hermitian variety in $PG(n,q^2)$.

Proof. We have to prove that all hyperplanes intersect the pivotted set in the correct number of points. Since we only replace points in the tangent space Π through p, we only have to look at the intersection of the hyperplanes α with Π.

1) If α equals Π, then α has the same number of intersection points with the pivotted set as with $H(n, q^2)$.

2) Next suppose that α intersects Π in an $(n-2)$-dimensional space. If α contains p then there are two cases to consider. The first possibility is that α intersects $H(n-2, q^2)$ and H' in the same number of points, in which case the total intersection number of α and the pivotted set is the same as the intersection number of α with $H(n, q^2)$. The second possibility is that α has different intersection numbers with $H(n-2, q^2)$ and H', but then this difference is equal to,

$$|H(n-3, q^2)| - |pH(n-4, q^2)| = \pm q^{n-3},$$

and so the total difference for the intersection size is $+/-q^2q^{n-3} = +/-q^{n-1}$ which equals

$$|H(n-1, q^2)| - |pH(n-2, q^2)|,$$

hence we get a valid intersection number.

If α does not contain p, then α intersects the intersection of the pivotted set and Π in a set of size $|H'| = |H(n-2, q^2)|$, hence the number of intersection points is unchanged. □

The second construction of a quasi-Hermitian variety only works in odd dimension since in even dimension the generators are too small for this construction to work. It is the Hermitian analogue of a theorem of Delanote [3].

Theorem 3
Let Π be an $(n-1)$-dimensional space lying on $H(2n+1, q^2)$. Consider the $q+1$ generators G_i, $1 \leq i \leq q+1$ on $H(2n+1, q^2)$ through Π. Consider also spaces Π_i, $1 \leq i \leq q+1$ through Π inside the tangent space Π^* of $H(2n+1, q^2)$ at Π which intersect the Hermitian variety exactly in Π. Consider

$$H' = (H(2n+1, q^2) \setminus \cup_i G_i) \cup (\cup_i \Pi_i)$$

This set H' is a quasi-Hermitian variety in $PG(2n+1, q^2)$.

Proof. Again we only have to look at the intersection of the hyperplanes α with Π^* since only there we replace points. If α contains Π^* then α has the same number of intersection points with H' as with $H(2n+1,q^2)$. So suppose that α intersects Π^* in an n-dimensional subspace.

1) If α intersects Π^* in one of the generators G_i, then it is a tangent hyperplane, hence we loose the correct number of points.

$$|pH(2n-1,q^2)| - |G\backslash\Pi| = |H(2n,q^2)|.$$

2) If $H \cap \Pi^*$ is one of the Π_i then H is a secant hyperplane, hence we have the correct number of points.

$$|\Pi_i\backslash\Pi| + |H(2n,q^2)| = |pH(2n-1,q^2)|.$$

3) The last possibility is that H intersects each of the $q+1$ n-dimensional spaces G_i in an $(n-1)$-dimensional space P_i with $P_i \cap \Pi = \alpha \cap \Pi = Y$ an $(n-2)$-dimensional space.

Let $\pi_j \cap \alpha = P_{q+1+j}$, with $j = 1, 2, ..., q+1$. We have replaced $(\cup_{j=1}^{q+1} P_j)\backslash Y$ by $(\cup_{j=q+2}^{2(q+1)} P_j)\backslash Y$. This clearly leaves the number of intersection points unchanged. □

Next we prove the Hermitian analogue of a remark in the Ph.D. thesis of Delanote [3]. Again we give a construction of a quasi-Hermitian variety, one which only works in odd dimension for $q = 2$.

Theorem 4
Consider $H' = H(2n+1,q^2)\backslash G$ where G is a generator of $H(2n+1,q^2)$. The complement of H' in $PG(2n+1,q^2)$ is a quasi-Hermitian variety in $PG(2n+1,q^2)$ if and only if $q = 2$.

Proof. A hyperplane α either contains G or intersects G in an $(n-1)$-dimensional space. If α contains G we know α is a tangent hyperplane. So the possible intersections of H' with hyperplanes are

$$|pH(2n-1,q^2)| - |G| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1},$$

$$|H(2n,q^2)| - |PG(n-1,q^2)| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1},$$

$$|pH(2n-1,q^2)| - |PG(n-1,q^2)| = \frac{q^{4n+1} - q^{2n+1}}{q^2 - 1} + q^{2n}.$$

So we get a two-character set. When looking at the complement of H' in $PG(2n+1,q^2)$ we get the following two intersection numbers with hyperplanes.

$$h_1 = \frac{q^{4n+2} - q^{4n+1} + q^{2n+1} - 1}{q^2 - 1}$$

$$h_2 = \frac{q^{4n+2} - q^{4n+1} + q^{2n+1} - q^{2n+2} + q^{2n} - 1}{q^2 - 1}$$

Hence we get the right intersection numbers if and only if $q = 2$. $\square$

3 Singular quasi-quadrics

First we recall the theorem of Bose and Burton.

Definition 5
A blocking set with respect to t-spaces *in $PG(n,q)$ is a set B of points such that every t-dimensional subspace of $PG(n,q)$ meets B in at least one point.*

The following result by Bose and Burton gives a nice characterization of the smallest ones [1].

Theorem 6
If B is a blocking set with respect to t-spaces in $PG(n,q)$ then $|B| \geq |PG(n-t,q)|$ and equality holds if and only if B is an $(n-t)$-dimensional subspace.

Consider a set $\mathcal{K}$ of q^3+q+1 points in $PG(4,q)$ such that every hyperplane intersects $\mathcal{K}$ in $q+1$, q^2+1 or q^2+q+1 points. A solid intersecting $\mathcal{K}$ in i points will be called an *i-solid.*

Theorem 7
If $q \neq 4$, then all $(q+1)$-solids contain a line which intersects $\mathcal{K}$ in at least q points. For all q, if there are at least 3 $(q+1)$-solids which intersect $\mathcal{K}$ in a full line, then the set $\mathcal{K}$ is a cone with vertex a point p and base an ovoid.

If we do not assume there are at least 3 $(q+1)$-solids which intersect $\mathcal{K}$ in a line, we have the following counterexamples:

Example 1
Let $q = 2$ and let $\mathcal{O}$ be an ovoid in a hyperplane Γ of $\mathrm{PG}(4,q)$. Let π be a tangent plane at $\mathcal{O}$ in Γ, say at the point x of $\mathcal{O}$. Let $p_1 \neq x$ and $p_2 \neq x$ be

two different points in π and consider two disjoint lines L_1 and L_2, through p_1 and p_2 respectively, which are not contained in Γ. Then the point set $\mathcal{K} = \mathcal{O} \cup L_1 \cup L_2$ satisfies all the desired intersection properties. **Remark.** *Placing the lines L_1 and L_2 in different positions yields other examples for the case $q = 2$.*

Example 2
Let $\mathcal{O}$ be an ovoid in a hyperplane Γ of $\mathrm{PG}(4, q)$, let p be a point not in Γ and consider the cone $\mathcal{K} := p\mathcal{O}$. Let π be a tangent plane at $\mathcal{O}$ in Γ, say at the point x of $\mathcal{O}$, and let L be a line in π through x. Then the set $\mathcal{K}' := \mathcal{K} \setminus px \cup L$ satisfies all the desired intersection properties.

We will prove Theorem 7 in several steps, which are described below.

Lemma 8
There are $(q^2 + 1)$ $(q + 1)$-solids.

Proof. Call the number of $(q+1)$-solids, (q^2+1)-solids and (q^2+q+1)-solids a, b and c respectively. Counting the total number of solids in a 4-space, the incident pairs (p, α) where p is a point of $\mathcal{K}$ and α a solid, and the number of ordered triples (p, r, α) where p and r are distinct points of $\mathcal{K}$ lying in the solid α respectively, yields the following equations

$$a + b + c = \frac{q^5 - 1}{q - 1},$$

$$a(q + 1) + b(q^2 + 1) + c(q^2 + q + 1) = (q^3 + q + 1)\frac{q^4 - 1}{q - 1},$$

$$a(q + 1)q + b(q^2 + 1)q^2 + c(q^2 + q + 1)(q^2 + q) = (q^3 + q + 1)(q^3 + q)\frac{q^3 - 1}{q - 1}.$$

Solving these equations completes the proof.

Lemma 9

(i) *Every plane which does not meet $\mathcal{K}$ is contained in exactly 2 $(q + 1)$-solids.*

(ii) *Two $(q + 1)$-solids intersect in at most one point of $\mathcal{K}$.*

(iii) *Any plane contains at most $2q + 1$ points of the set $\mathcal{K}$.*

(iv) *All $(q+1)$-solids which intersect $\mathcal{K}$ in a line have a point of $\mathcal{K}$ in common.*

Proof. Consider a plane π and suppose that $|\pi \cap \mathcal{K}| = x$. Consider all solids through π in $PG(4,q)$ and denote the number of them which are $(q+1)$-solids, (q^2+1)-solids and (q^2+q+1)-solids by a, b and c respectively. This yields the following equation:

$$x + a(q+1-x) + b(q^2+1-x) + c(q^2+q+1-x) = q^3+q+1.$$

After simplifying we get $\alpha + \gamma - x = (\alpha - 1)q$. This proves (i), (ii) and (iii) immediately. Hence, all $(q+1)$-solids which intersect $\mathcal{K}$ in a line have a point in common, otherwise we get a plane intersecting $\mathcal{K}$ in at least $3q$ points. $\square$

Lemma 10
Every $(q+1)$-solid contains a line which intersects $\mathcal{K}$ in at least q points.

Proof.
Let Σ be a $(q+1)$-solid, and let L be any line of Σ having non-trivial intersection with $\mathcal{K}$. Suppose that L intersects $\mathcal{K}$ in $1+k$ points. We calculate a lower bound for the number of exterior planes (i.e. not intersecting $\mathcal{K}$) of Σ. One easily sees there are at least $(q-k)(q^2+k)$ exterior lines intersecting L. Furthermore, on each such line there are at least k exterior planes. Since every exterior plane intersects L it contains exactly $q+1$ of exterior lines intersecting L. It follows there are at least

$$E = \frac{k(q-k)(q^2+k)}{q+1}$$

exterior planes in Σ. By (i) of Lemma 9 this implies there at least $E+1$ $(q+1)$-solids in PG$(4,q)$. Hence $E \leq q^2$ must hold. We obtain that $k \in \{0, 1, q-1, q\}$ or $(k,q) = (2,4)$. We will deal with $(k,q) = (2,4)$ at the end of the proof. So suppose that $k \in \{0, 1, q-1, q\}$, and that $\Sigma \cap \mathcal{K}$ would be an arc in Σ. Let π be a plane of Σ intersecting $\mathcal{K}$ in $l+1$ points. We may assume without loss of generality that $l \geq 2$. In π there are exactly $q^2+q+1-(q+1-l)-(l+1)l/2$ lines exterior to $\mathcal{K}$, and through each of these lines there pass at least l planes of Σ exterior to $\mathcal{K}$. As the total number of exterior planes in Σ can be at most q^2 it follows that $q^2 \leq l^2/2$, a contradiction.

We now deal with the case $(k,q) = (2,4)$. There is a line L containing 3 points of $\Sigma \cap \mathcal{K}$. Let M be the line spanned by the two remaining points in $\Sigma \cap \mathcal{K}$.

(a) If $L \cap M = \emptyset$ then inside π there are 6 lines exterior to $\mathcal{K}$. Hence in Σ there are 18 planes exterior to $\mathcal{K}$. By (i) of Lemma 9 each of them is contained in two $(q+1)$-solids. This yields a contradiction since there are only 17 $(q+1)$-solids by Lemma 8.

(b) If $L \cap M$ is a point $p \notin \mathcal{K}$ then inside π there are 5 lines exterior to $\mathcal{K}$. Hence in Σ there are 20 planes exterior to $\mathcal{K}$. By (i) of Lemma 9 each of them is contained in two $(q+1)$-solids. This yields a contradiction since there are only 17 $(q+1)$-solids by Lemma 8.

(c) If $L \cap M$ is a point $p \in \mathcal{K}$ then inside π there are 4 lines exterior to $\mathcal{K}$. Hence in Σ there are 16 planes exterior to $\mathcal{K}$. By (i) of Lemma 9 each of them is contained in two $(q+1)$-solids. First, if we assume at least one $(q+1)$-solid intersects $\mathcal{K}$ in a line, we obtain a contradiction since there are only 17 $(q+1)$-solids by Lemma 8.

So we may suppose there are no $(q+1)$-solids intersecting $\mathcal{K}$ in a full line. Hence every $(q+1)$-solids contains either 1 line intersecting $\mathcal{K}$ in 4 points (type I) or 2 lines intersecting $\mathcal{K}$ each in 3 points (type II). From the above, the intersection of two $(q+1)$-solids of type II can never contain a point of $\mathcal{K}$ (as there are 16 exterior planes in such $(q+1)$-solids). Hence there must be $(q+1)$-solids of type I. Let Π_1 be such solid, and call h_1 the unique point on L_1 not belonging to $\mathcal{K}$, where L_1 is the unique line of Π_1 intersecting $\mathcal{K}$ in 4 points. We immediately see that h_1 is contained in at least 12 $(q+1)$-solids. Furthermore, as all exterior planes of Π_1 pas through h_1, h_1 is contained in all solids of type II. Now define Π_2 and h_2 analogously to Π_1 and h_1 for a second $(q+1)$-solid Π_2 of type I. Assume that $h_1 \neq h_2$. Since also h_2 is contained in at least 12 $(q+1)$-solids it follows that the line h_1h_2 is contained in at least 7 $(q+1)$-solids. This implies the existence of a plane through h_1h_2 containing at least 3 $(q+1)$-solids, a contradiction. Hence $h_1 = h_2$, and the point h_1 is contained in all $(q+1)$-solids. Now consider any $(q+1)$-solid Σ of type II. Then $h_1 \in \Sigma$, and furthermore every exterior plane of Σ must contain h_1. This is clearly impossible.

□

Lemma 11
All $(q+1)$-solids which intersect $\mathcal{K}$ in a line have a point in common. Suppose there are at least three different $(q+1)$-solids which intersect $\mathcal{K}$ in a line.

Then all $(q+1)$-solids intersect $\mathcal{K}$ in a line.

Proof. Clearly all lines L_i which are the intersections of $(q+1)$-solids and $\mathcal{K}$ have to intersect each other and moreover they all have a point p in common, otherwise there is a plane which intersects $\mathcal{K}$ in more than $2q+1$ points, a contradiction. Suppose there is a solid Π_3 which does not contain p. The space Π_3 intersects each of the lines L_i in a point of $\mathcal{K}$. Since there are at least 3 such lines we get a plane containing more than $2q+1$ points of $\mathcal{K}$, a contradiction. Consider an arbitrary plane π which intersects $\mathcal{K}$ in x points not through p and consider all solids through it. Then we get at least $q+1+q(q^2+1-x)$ points in $\mathcal{K}$. Hence π is blocked by $\mathcal{K}$. By Theorem 5 this implies that all $(q+1)$-solids intersect $\mathcal{K}$ in a line. $\square$

Now we can complete the proof of Theorem 7.

Proof. By Lemma 11 all q^2+1 $(q+1)$-solids intersect $\mathcal{K}$ in a line, and these lines have a point p in common. Since $1+q(q^2+1)=q^3+q+1$, it follows that p is collinear with all other points of the set $\mathcal{K}$.

Let M be a line not through p containing at least three points of $\mathcal{K}$, say r, s and t. Then t is contained in the plane π spanned by the lines $\langle p, r\rangle$ and $\langle p, s\rangle$. Hence π intersects $\mathcal{K}$ in at least $2q+2$ points, a contradiction by (iii) of Lemma 9. Hence, all lines not through p intersect $\mathcal{K}$ in at most 2 points. Consider a solid Π not through p. Then, since $q>2$ and by the above, Π intersect $\mathcal{K}$ in an ovoid. $\square$

References

[1] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combin. Theory*, 1:96–104, 1966.

[2] F. De Clerck, N. Hamilton, C. O'Keefe, T. Penttila Quasi-quadrics and related structures, *Australas. J. Combin.* **22** (2000), 151-166.

[3] M. Delanote Constructions and characterisations of (semi)partial geometries, Ph.D Thesis, Ghent University, 2001.

[4] S. De Winter and J. Schillewaert. A characterization of finite polar spaces by intersection numbers. *Combinatorica*, submitted.

[5] N . Durante, V. Napolitano and D. Olanda On quadrics of PG$(3, q)$, preprint, 2006.