

# Flat Laguerre planes of Kleinewillinghöfer type III.B

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## Abstract

Kleinewillinghöfer classified Laguerre planes with respect to *central automorphisms*. Polster and Steinke investigated flat Laguerre planes and their so-called Kleinewillinghöfer types, that is, the Kleinewillinghöfer types with respect to the full automorphism group. For some of these types the existence question remained open. We provide strong necessary existence conditions for flat Laguerre planes of Kleinewillinghöfer type III.B and provide examples of such planes of types III.B.1 and III.B.3.

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## 1 Introduction

Similar to the Lenz-Barlotti classification of projective planes, compare [11], Anhang, Section 6, Kleinewillinghöfer [6] classified Laguerre planes with respect to *central automorphisms*, that is, permutations of the point set of the Laguerre plane such that parallel classes are mapped to parallel classes and circles are mapped to circles and such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. In [14] and [19] flat Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types with respect to the full automorphism group. In particular, all possible types of flat Laguerre planes with respect to Laguerre translations, were completely determined in [14] and the case of Laguerre homologies was dealt with in [19]. Examples for some of the possible Kleinewillinghöfer types of flat Laguerre planes can be found in [14] Section 6, [18] and [10].

In this paper we provide examples for flat Laguerre planes of Kleinewillinghöfer types III.B.1 and III.B.3 thus completely covering type III together with the models from [14], Section 6. Hence only the existence of three combined types remains open.

## 2 Flat Laguerre Planes

A *flat* or *2-dimensional Laguerre plane*  $\mathcal{L} = (Z, \mathcal{C})$  is an incidence structure of points and circles whose point set is the cylinder  $Z = \mathbb{S}^1 \times \mathbb{R}$  (where  $\mathbb{S}^1$  usually is represented as  $\mathbb{R} \cup \{\infty\}$ ), whose circles  $C \in \mathcal{C}$  are graphs of continuous functions  $\mathbb{S}^1 \rightarrow \mathbb{R}$  such that

any three points no two of which are on the same generator  $\{c\} \times \mathbb{R}$  of the cylinder can be joined by a unique circle and such that the circles which touch a fixed circle  $C$  at  $p \in C$ , that is,  $C$  itself and the circles which have only  $p$  in common with  $C$ , partition the complement in  $Z$  of the generator that contains  $p$ . For more information on flat Laguerre planes we refer to [2] and [4] or [13] Chapter 5. We say that two points of  $Z$  are parallel if they are on the same generator of  $Z$ . This defines an equivalence relation on  $Z$  whose equivalence classes, also called parallel classes, are the generators of  $Z$ .

It readily follows that for each point  $p$  of  $\mathcal{L}$  the incidence structure  $\mathcal{A}_p = (A_p, \mathcal{L}_p)$  whose point set  $A_p$  consists of all points of  $\mathcal{L}$  that are not parallel to  $p$  and whose line set  $\mathcal{L}_p$  consists of all restrictions to  $A_p$  of circles of  $\mathcal{L}$  passing through  $p$  and of all parallel classes not passing through  $p$  is an affine plane, which extends to a projective plane  $\mathcal{P}_p$ . We call  $\mathcal{A}_p$  and  $\mathcal{P}_p$  the *derived affine* or *projective plane at  $p$* , respectively. In fact, the geometric axioms of a Laguerre plane are equivalent to all derived incidence structures  $\mathcal{A}_p$  as defined above being affine planes.

The *classical flat Laguerre plane* is obtained as the geometry of non-trivial plane sections of a cylinder in  $\mathbb{R}^3$  with an ellipse in  $\mathbb{R}^2$  as base, or equivalently, as the geometry of non-trivial plane sections of an elliptic cone, in real 3-dimensional projective space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in the construction of the classical flat Laguerre plane by arbitrary ovals in  $\mathbb{R}^2$ , i.e., convex, differentiable simply closed curves, we also obtain flat Laguerre-planes. These are the so-called *ovoidal flat Laguerre planes*.

Every automorphism of a flat Laguerre plane is continuous and thus a homeomorphism of  $Z$ . The collection of all automorphisms of a flat Laguerre plane  $\mathcal{L}$  forms a group with respect to composition, the automorphism group  $\Gamma$  of  $\mathcal{L}$ . This group is a Lie group of dimension at most 7 with respect to the compact-open topology; see [16]. We call the dimension of  $\Gamma$  the *group dimension* of  $\mathcal{L}$ . The maximum dimension is attained precisely in the classical flat Laguerre plane. In fact, group dimension at least 6 implies classical. Furthermore, flat Laguerre planes of group dimension 5 must be special ovoidal Laguerre planes; see [9] Theorem 1.

If the Lie group  $G$  acts on a manifold  $M$ , then we get the following dimension formula  $\dim G = \dim G_p + \dim G(p)$  where  $G_p$  and  $G(p)$  are the stabilizer and orbit, respectively, of the point  $p \in M$ .

### 3 Kleinewillinghöfer types of flat Laguerre planes

Kleinewillinghöfer considered four kinds of central automorphisms:  $C$ -homologies,  $G$ -translations,  $(G, B(q, C))$ -translations and  $(p, q)$ -homotheties; see the following for definitions. Central automorphisms are automorphisms that have at least one fixed point and induce central collineations in the derived projective plane at this fixed point. The four different kinds of central automorphisms above are distinguished according to the relative position of centre and axis and whether or not the axis is the line at infinity of the derived affine plane at one of its fixed points. The notations of translation, homothety

and homology describe the sort of central collineation one sees in this derived affine plane.

A subgroup of central automorphisms that have the same ‘centre’ and ‘axis’ is *linearly transitive* if the induced group of central collineations in a derived projective plane at one of the fixed points is transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. Kleinewillinghöfer considered groups of automorphisms and determined their types according to linearly transitive subgroups of central automorphisms contained in them. In particular, a group of automorphisms is said to be linearly transitive if it contains a linearly transitive subgroup of central automorphisms. In case of the full automorphism group  $\Gamma$  we then say that the Laguerre plane is of the corresponding Kleinewillinghöfer type of  $\Gamma$ .

A *Laguerre homology* of a Laguerre plane  $\mathcal{L}$  is an automorphism of  $\mathcal{L}$  that is either the identity or fixes precisely the points of one circle. One speaks of a *C-homology* if  $C$  is the circle that is fixed. A *C-homology* induces a homology of the derived projective plane  $\mathcal{P}_q$  at each  $q \in C$  with infinite centre  $\omega$ . With respect to Laguerre homologies Kleinewillinghöfer obtained seven types of Laguerre planes, labelled I, II, III, IV, V, VI and VII; see [6], Satz 3.1. Of these types type VI cannot occur in flat Laguerre planes; see [14] Proposition 3.4.

A *Laguerre translation* of  $\mathcal{L}$  is an automorphism of  $\mathcal{L}$  that is either the identity or fixes precisely the points of one parallel class and induces a translation in the derived affine plane at one of its fixed points. Laguerre translations. come in two different varieties. Firstly, a non-identity *G-translation* of  $\mathcal{L}$  is a Laguerre translation that fixes precisely the points of the parallel class  $G$  and furthermore fixes each parallel class globally. For the second variety of Laguerre translations one uses a tangent bundle  $B(p, C)$ , that is, all circles that touch the circle  $C$  at the point  $p$ . In the derived affine plane at  $p$  the tangent bundle represents a parallel class of lines, and we can look at translations in this direction. Then a  $(G, B(p, C))$ -translation of  $\mathcal{L}$  is a Laguerre translation that fixes  $C$  (and each circle in  $B(p, C)$ ) globally. With respect to Laguerre translations Kleinewillinghöfer obtained 11 types of Laguerre planes, labelled A through to K; see [6] Satz 3.3, or [7] Satz 2. Of these types the types F, I and J cannot occur in flat Laguerre planes; see [14] Proposition 4.8.

Finally, a *Laguerre homothety* of  $\mathcal{L}$  is an automorphism of  $\mathcal{L}$  that is either the identity or fixes precisely two non-parallel points and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a  $\{p, q\}$ -homothety if  $p, q$  are the two fixed points. With respect to Laguerre homotheties Kleinewillinghöfer [6] Satz 3.2, or [7] Satz 1, obtained 13 types of Laguerre planes, labelled 1 through to 13. Types 5, 6, 7, 9, 10 and 12 cannot occur in flat Laguerre planes; see [14] Proposition 5.6 and [19].

Combining all three classifications Kleinewillinghöfer obtained a total of 46 combined types. Of these types 21 combined types cannot occur in flat Laguerre planes. There are models of flat Laguerre planes of types I.A.1, I.B.1, I.B.3, I.C.1, I.E.1, I.E.4, I.G.1, I.H.1, I.H.11, II.A.1, II.E.1, II.E.4, II.G.1, III.H.1, III.H.11, IV.A.1, IV.A.2, VII.D.1, VII.D.8 and VII.K.13; see [14] Section 6, [10], [18], [19], [20]. Here a combined type just refers to the respective single types. E.g., type III.B.3 refers to type III with respect to Laguerre homotheties, type B with respect to Laguerre translations, and type 3 with respect to

Laguerre homotheties. Note that there is a flat Laguerre plane of each of the single Kleinewillinghöfer types not excluded for flat Laguerre planes, except for type V with respect to Laguerre homologies.

In particular, the Kleinewillinghöfer types III.B.1 and III.B.3 we are interested in in this paper are defined as follows. In type III (with respect to Laguerre homologies) the set  $\mathcal{Z}$  of all circles  $C$  for which the automorphism group of the flat Laguerre plane  $\mathcal{L}$  is linearly transitive (with respect to  $C$ -homologies) consists of a tangent bundle  $B(p, C)$  of  $\mathcal{L}$ . In type B (with respect to Laguerre translations) there is no tangent bundle for which the group of Laguerre translations is linearly transitive and exactly one parallel class  $G$  for which the group of  $G$ -translations is linearly transitive. In type 3 (with respect to Laguerre homotheties) there are a point  $p$  and a parallel class  $G$  with  $p \notin G$  such that each group of  $\{p, q\}$ -homotheties is linearly transitive for all  $q \in G$ . A flat Laguerre plane of type III.B must be of type III.B.1 or III.B.3.

With the models in this paper only the existence of combined types I.A.2, II.A.2 and V.A.1 remains open in flat Laguerre planes.

## 4 Some necessary conditions for a flat Laguerre plane of type III.B.3

Let  $\mathcal{L}$  be a flat Laguerre plane of type III.B.3. Then the derived affine plane at the distinguished point  $p$  as in types III and 3 with respect to Laguerre homologies and Laguerre homotheties must be Desarguesian by the Lenz-Barlotti classification of projective planes, compare [11], Anhang, Section 6. We introduce coordinates such that the distinguished point is  $(\infty, 0)$  and the distinguished parallel class is  $\Pi_0 = \{0\} \times \mathbb{R}$ . Since  $\mathcal{A}_{(\infty, 0)}$  is Desarguesian, the circles through  $(\infty, 0)$  are the extended Euclidean lines

$$C_{0,b,c} = \{(x, bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, 0)\}$$

and the group generated by all the central automorphisms as in types III, B and 3 consists of the transformations

$$\varphi_{r,s,t} : (x, y) \mapsto (rx, sy + t)$$

for  $r, s, t \in \mathbb{R}$ ,  $r, s \neq 0$ , suitably extended onto the parallel class  $\Pi_\infty = \{\infty\} \times \mathbb{R}$  at infinity.

The automorphism group  $\Gamma$  of  $\mathcal{L}$  must fix the distinguished point  $p = (\infty, 0)$ , the parallel class  $\Pi_0$  and the tangent bundle as in type III. Each  $\gamma \in \Gamma$  induces a collineation of  $\mathcal{A}_p$  that fixes two points on the line at infinity (the points at infinity on vertical lines and on lines that come from circles in the distinguished tangent bundle) and a vertical line. Hence  $\Gamma$  is at most 3-dimensional by the dimension formula, see the end of section 2. On the other hand the collection of automorphisms  $\varphi_{r,s,t}$  is 3-dimensional so that  $\Gamma$  must be 3-dimensional. In fact,

$$\Gamma = \{\varphi_{r,s,t} \mid r, s, t \in \mathbb{R}, r, s \neq 0\}.$$

The stabiliser  $H$  of the circle  $C_{0,0,0}$  is

$$H = \{\varphi_{r,s,0} \mid r, s \neq 0\}.$$

Since  $H$  is a 2-dimensional abelian group and because the subgroups of all homologies or all homotheties are transitive on  $\Pi_\infty^* = \Pi_\infty \setminus \{(\infty, 0)\}$ , there is a 1-dimensional subgroup of  $H$  that acts trivially on  $\Pi_\infty$ . The connected component containing the identity of such a subgroup must be of the form  $\{\varphi_{r,r^k,0} \mid r > 0\}$  for some  $k \neq 0$ . Indeed, suppose that  $s = \phi(r)$  for some continuous function  $\phi : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ . Since this component is a subgroup,  $\phi$  satisfies the functional equation  $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$ , which yields the above, see also section A.1.4 of [13]. Since the homology  $\varphi_{1,-1,0}$  and the homothety  $\varphi_{-1,-1,0}$  act orientation reversing on  $\Pi_\infty$ , we see that  $\varphi_{-1,1,0}$  is orientation preserving on  $\Pi_\infty$ . But  $\varphi_{-1,1,0}$  is also involutory so that  $\varphi_{-1,1,0}$  must act trivially on  $\Pi_\infty$ . Composing this automorphism with all automorphisms in  $\{\varphi_{r,r^k,0} \mid r > 0\}$  for some  $k \neq 0$  shows that

$$K = \{\varphi_{r,|r|^k,0} \mid r \neq 0\}$$

is in the kernel of the action of  $H$  on  $\Pi_\infty$ .

Let  $C_1$  be the circle through  $(1, 1)$  that touches  $C_{0,0,0}$  at  $(0, 0)$ . Without loss of generality we may assume that  $C_1$  also passes through the point  $(\infty, 1)$ . Since  $C_1$  is fixed by  $K$ , we obtain a functional equation for the function  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$  describing  $C_1$  (that is,  $C_1 = \{(x, f_1(x)) \mid x \in \mathbb{R}\} \cup \{(\infty, 1)\}$ ), which results in  $f_1$  being of the form  $f_1(x) = |x|^k$  for some  $k \neq 0$ . Since  $C_1$  induces in the derived projective plane at  $(\infty, 0)$  an oval that has the line at infinity as a tangent, the curve  $y = f_1(x)$  must be parabolic. Parabolic curves are characterized by  $f_1$  (or  $-f_1$ ) being strictly convex and  $\lim_{x \rightarrow +/\infty} (f_1(x) - cx) = +\infty$  for all  $c \in \mathbb{R}$ , compare [13], Section 5.3.1. Hence, we obtain that  $k > 1$ . Applying the automorphisms  $\phi_{(1,a,c)}$  to  $C_1$  shows that  $\mathcal{L}$  has the circles

$$C_{a,b,c} = \{(x, a|x|^k + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$$

for  $a, b, c \in \mathbb{R}$ ,  $ab = 0$ . (The orbit of  $C_1$  plus the circle through  $(\infty, 0)$ .) Moreover, since  $\varphi_{(r,s,t)}$  sends  $(x, a|x|^k + bx + c)$  to  $(rx, s(a|x|^k + bx + c) + t)$ , putting  $x' = rx$  shows that  $\varphi_{r,s,t}$  acts on  $\Pi_\infty$  by

$$\varphi_{r,s,t}(\infty, y) = (\infty, sy/|r|^k).$$

$\Gamma$  has four orbits on the cylinder  $Z$ , namely,  $\{(\infty, 0)\}$ ,  $\Pi_\infty \setminus \{(\infty, 0)\}$ ,  $\Pi_0$  and  $Z \setminus (\Pi_\infty \cup \Pi_0)$ . Furthermore,  $\Gamma$  has one orbit on the set of all remaining circles (not of the form  $C_{a,b,c}$  with  $ab = 0$  as above). Let  $C_2$  be the circle through  $(\infty, 1)$  that touches  $C_{0,1,0}$  at  $(0, 0)$  and let  $f_2$  be a function describing  $C_2$  so that  $C_2 = \{(x, f_2(x)) \mid x \in \mathbb{R}\} \cup \{(\infty, 1)\}$ . Applying all the transformations  $\varphi_{r,s,t}$ , we thus obtain the circles  $C_{r,s,t}^2 = \{(x, sf_2(x/r) + t) \mid x \in \mathbb{R}\} \cup \{(\infty, s|r|^k)\}$ .

Note that coherence in flat Laguerre planes implies that the circle through  $(\infty, 1)$  that touches  $C_{0,b,0}$  at  $(0, 0)$  tends to  $C_{1,0,0}$  as  $b$  tends to 0. Thus if  $s = |r|^k$  and  $s/r = r|r|^{k-2}$  tends to 0, that is,  $r \rightarrow 0$ , then  $|r|^k f_2(x/r)$  tends to  $|x|^k$ . But this implies that  $\lim_{x \rightarrow \infty} f_2(x)/|x|^k = 1$ .

In order to find a flat Laguerre plane of type III.B.1 one can look for planes which have ‘almost’ type III.B.3, that is, flat Laguerre planes for which the group of all  $\{(\infty, 0), (0, 0)\}$ -homotheties has four orbits (instead of three under full linear transitivity) on  $C_{0,0,0}$ , namely,  $\{(\infty, 0)\}$ ,  $\{(0, 0)\}$ ,  $\{(x, 0) \mid x > 0\}$  and  $\{(x, 0) \mid x < 0\}$ . This means that  $\Gamma$  is still 3-dimensional and that all  $\varphi_{r,s,t}$  for  $r, s, t \in \mathbb{R}$ ,  $r > 0$ ,  $s \neq 0$ , are automorphisms of the plane but  $\gamma_{-1,-1,0}$  is not. Hence most of the conditions about the structure of the plane obtained above remain valid. Such a construction can be achieved by pasting together along  $\Pi_\infty \cup \Pi_0$  two halves of flat Laguerre planes of type III.B.3 or by making sure that  $f_1$  or  $f_2$  is not symmetric.

## 5 The Models

In this section we present models for flat Laguerre planes of Kleinewillinghöfer types III.B.1 and III.B.3. According to the previous section all we have to do for type III.B.3 is specify  $k$  and the function  $f_2$  and then verify the axioms of a flat Laguerre plane. We choose a suitable polynomial of low degree for the function  $f_2$ . Since  $k = 2$  yields the classical Laguerre plane and because polynomials of odd degree are not parabolic, the lowest degree suitable is  $k = 4$ . Specifically, in the notation of the previous section, we use  $f_2(x) = x^4 + x^3 + x^2 + x$ . Then the limit condition on  $f_2$  is automatically satisfied. In order to simplify the formulas and avoid cubic roots we make a slight change to the way circles are parametrised. For  $a, b \in \mathbb{R}$  let

$$h_{a,b}(x) = a^3x^4 + a^2bx^3 + ab^2x^2 + b^3x$$

so that  $h_{a,b}(x) = a^3b^4f_2(x/b)$ . Then  $h_{a,b}$  is a strictly convex or strictly concave function of  $\mathbb{R}$  unless  $a = 0$ . Note that

$$\begin{aligned} h_{b,a}(x) &= x^5h_{a,b}(1/x) & x \neq 0 \\ h_{a,-b}(x) &= h_{a,b}(-x) \\ h_{a,b}(x) &= a^3h_{1,b/a}(x) & a \neq 0 \end{aligned}$$

For type III.B.1 the pasting along  $\Pi_\infty \cup \Pi_0$  is achieved with a fixed parameter  $q > 0$ . We define the sets

$$C_{a,b,c} = \{(x, h_{a,b}(x) + c) \mid x \geq 0\} \cup \{(x, qh_{a,b}(x) + c) \mid x < 0\} \cup \{(\infty, a^3)\}$$

for  $a, b, c \in \mathbb{R}$ . Explicitly, we have the following incidence structures.

### Description of the models $\mathcal{L}(q)$ .

Let  $q > 0$ . The incidence structure  $\mathcal{L}(q)$  has point set  $Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ ; two points  $(x_1, y_1), (x_2, y_2) \in Z$  are parallel if and only if  $x_1 = x_2$ . The sets

$$\begin{aligned} C_{a,b,c} &= \{(x, a^3x^4 + a^2bx^3 + ab^2x^2 + b^3x + c) \mid x \geq 0\} \\ &\quad \cup \{(x, q(a^3x^4 + a^2bx^3 + ab^2x^2 + b^3x) + c) \mid x < 0\} \cup \{(\infty, a^3)\}. \end{aligned}$$

for  $a, b, c \in \mathbb{R}$  are the circles of  $\mathcal{L}(q)$ .

We claim that  $\mathcal{L}(q)$  is a flat Laguerre plane with the collection of the above sets  $C_{a,b,c}$  as the circle set. For  $r, s, t \in \mathbb{R}$ ,  $r, s \neq 0$ , let the permutation  $\gamma_{r,s,t} : Z \rightarrow Z$  be defined by

$$\gamma_{r,s,t} : (x, y) \mapsto \begin{cases} (rx, s^3y + t), & \text{for } x \in \mathbb{R} \\ (\infty, s^3y/r^4), & \text{for } x = \infty \end{cases}.$$

It is readily verified that  $\gamma_{r,s,t}$  is an automorphism of  $\mathcal{L}(q)$  such that

$$\gamma_{r,s,t}(C_{a,b,c}) = C_{as/r^4/3, bs/r^{1/3}, s^3c+t}$$

for  $r, s, t \in \mathbb{R}$ ,  $r > 0$ ,  $s \neq 0$ . Furthermore,  $\sigma = \gamma_{-1,1,0}$  is an isomorphism from  $\mathcal{L}(q)$  to  $\mathcal{L}(1/q)$ . Note that  $\sigma$  is an automorphism of  $\mathcal{L}(1)$ .

Let

$$G = \{\gamma_{r,s,t} \mid r, s, t \in \mathbb{R}, s \neq 0, r > 0\}.$$

This is a group of automorphisms of  $\mathcal{L}(q)$  that has five orbits on the cylinder  $Z$ , namely  $\{(\infty, 0)\}$ ,  $\Pi_\infty \setminus \{(\infty, 0)\}$ ,  $\Pi_0$ ,  $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$  and  $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ .

**PROPOSITION 5.1** *The derived incidence structure  $\mathcal{A}_{(\infty,0)}$  of  $\mathcal{L}(q)$  at  $(\infty, 0)$  is the real Desarguesian affine plane.*

*Proof.* The non-vertical lines in  $\mathcal{A}_{(\infty,0)}$  are the traces on  $\mathbb{R}^2$  of circles  $C_{0,b,c}$  for  $b, c \in \mathbb{R}$ . After the coordinate transformation  $(x, y) \mapsto \begin{cases} (x, y), & \text{for } x \geq 0, \\ (qx, y), & \text{for } x < 0 \end{cases}$  we obtain the Euclidean lines  $\{(x, b^3x + c) \mid x \in \mathbb{R}\}$ .  $\square$

For the derived incidence structures at points  $(\infty, a)$ ,  $a \neq 0$ , on the parallel class at infinity it suffices to consider  $a = 1$  because all other points can be obtained by applying automorphisms of the form  $\gamma_{1,s,0}$ .

**PROPOSITION 5.2** *The derived incidence structure  $\mathcal{A}_{(\infty,1)}$  of  $\mathcal{L}(q)$  at  $(\infty, 1)$  is a non-Desarguesian affine plane.*

*Proof.* The non-vertical lines in  $\mathcal{A}_{(\infty,1)}$  are the traces on  $\mathbb{R}^2$  of circles  $C_{1,b,c}$  for  $b, c \in \mathbb{R}$ . Applying the coordinate transformation

$$(x, y) \mapsto \begin{cases} (x, y - x^4), & \text{for } x \geq 0 \\ (x, y - qx^4), & \text{for } x < 0 \end{cases}$$

we obtain an isomorphic incidence structure  $\mathcal{A}_1$  whose non-vertical lines are the sets

$$L_{b,c} = \{(x, bx^3 + b^2x^2 + b^3x + c) \mid x \geq 0\} \cup \{(x, q(bx^3 + b^2x^2 + b^3x) + c) \mid x < 0\}$$

for  $b, c \in \mathbb{R}$ .

We first show that two lines  $L_{b,c}$  and  $L_{b',c'}$  are parallel (that is, disjoint or identical) if and only if  $b = b'$  and that two non-parallel lines intersect in precisely one point. Clearly the lines are parallel if  $b = b'$ . Now assume that  $b \neq b'$ . Then

$$(bx^3 + b^2x^2 + b^3x) - (b'x^3 + (b')^2x^2 + (b')^3x) = (b - b')(x^3 + (b + b')x^2 + (b^2 + bb' + (b')^2)x)$$

and the cubic polynomial

$$p(x) = x^3 + (b + b')x^2 + (b^2 + bb' + (b')^2)x$$

is strictly increasing. Hence  $p(x) = \frac{c'-c}{b-b'}$  has exactly one solution  $x_+$ . This solution gives rise to a point of intersection if  $x_+ \geq 0$ . Similarly,  $p(x) = \frac{c'-c}{q(b-b')}$  has exactly one solution  $x_-$ , which gives rise to a point of intersection if  $x_- < 0$ . However,  $p(0) = 0$  and  $q > 0$  so that  $x_+$  and  $x_-$  are either both 0 or have the same sign. In the latter case precisely one of  $x_+$  or  $x_-$  corresponds to a point of intersection of  $L_{b,c}$  and  $L_{b',c'}$ . This shows that  $L_{b,c}$  and  $L_{b',c'}$  are not parallel and indeed have precisely one point in common. Obviously, a vertical line and a line  $L_{b,c}$  also have precisely one point in common.

Next we verify the axiom of joining for an affine plane. So let  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  be two distinct points. If  $x_1 = x_2$ , then the only line in  $\mathcal{A}_1$  joining these two points is the vertical line  $x = x_1$ . We now assume that  $0 \leq x_1 < x_2$ . In this case we obtain the system of equations

$$\begin{aligned} y_1 &= bx_1^3 + b^2x_1^2 + b^3x_1 + c \\ y_2 &= bx_2^3 + b^2x_2^2 + b^3x_2 + c \end{aligned}$$

for the parameters  $b$  and  $c$  of a joining line  $L_{b,c}$ . Taking the difference of these equations and dividing by  $x_2 - x_1 \neq 0$ , we find that the parameter  $b$  is determined by

$$b^3 + (x_1 + x_2)b^2 + (x_1^2 + x_1x_2 + x_2^2)b - \frac{y_2 - y_1}{x_2 - x_1} = 0,$$

which has precisely one root. (This basically is the polynomial  $p$  from above.) From  $b$  we obtain  $c = y_1 - (bx_1^3 + b^2x_1^2 + b^3x_1)$ , and the joining line  $L_{b,c}$  is uniquely determined. The case  $x_1 < x_2 \leq 0$  is dealt with similarly or we can use the isomorphism  $\sigma$  and obtain the desired result from the previous case in  $\mathcal{L}(1/q)$ .

In the last case to be considered we assume that  $x_1 < 0 < x_2$ . We then solve the system of equations

$$\begin{aligned} y_1 &= bx_1^3 + b^2x_1^2 + b^3x_1 + c \\ y_2 &= qx_2^3 + qb^2x_2^2 + qb^3x_2 + c \end{aligned}$$

for  $b$  and  $c$ . Taking the difference of these equations yields

$$(qx_2 - x_1)b^3 + (qx_2^2 - x_1^2)b^2 + (qx_2^3 - x_1^3)b + y_1 - y_2 = 0.$$



The leading coefficient of this cubic polynomial in  $b$  is positive and the above equation has at least one root. Each such root  $b$  and  $c = y_1 - (bx_1^3 + b^2x_1^2 + b^3x_1)$  results in a joining line  $L_{b,c}$ . However, as seen before, two distinct lines intersect in at most one point and so  $b$  and  $c$ , and thus  $L_{b,c}$ , must be unique. This shows that  $\mathcal{A}_1$  is a linear space.

We now turn to the parallel axiom in  $\mathcal{A}_1$ . Since the parallel axiom is clearly satisfied for vertical lines, let a point  $(x_0, y_0)$  and a non-vertical line  $L_{b,c}$  be given. A line parallel to  $L_{b,c}$  must be of the form  $L_{b,c'}$  for some  $c' \in \mathbb{R}$ . The line parameter  $c'$  is uniquely determined by

$$c' = \begin{cases} y_0 - (bx_0^3 + b^2x_0^2 + b^3x_0), & \text{for } x_0 \geq 0 \\ y_0 - q(bx_0^3 + b^2x_0^2 + b^3x_0), & \text{for } x_0 < 0 \end{cases}.$$

This shows that  $\mathcal{A}_1$  (and thus  $\mathcal{A}_{(\infty,1)}$ ) is an affine plane.

We finally show that  $\mathcal{A}_1$  is non-Desarguesian. To see this consider the two triangles with vertices  $p_1 = (0, 0)$ ,  $p_2 = (1, 3)$ ,  $p_3 = (1, -1)$  and  $p'_1 = (u, u^2 - 3)$ ,  $p'_2 = (2, 3)$ ,  $p'_3 = (2, -1)$ , respectively, where  $u$  is the root of  $X^3 + X - 8 = 0$  (that is,  $u = \frac{1}{3}\sqrt[3]{108 + \sqrt{1299}} - \frac{1}{\sqrt[3]{108 + \sqrt{1299}}} \approx 1.833751$ ). Then corresponding sides  $p_i p_j$  and  $p'_i p'_j$  of the triangles are parallel (these are the lines  $L_{1,0}$  and  $L_{1,-11}$  for  $i = 1, j = 2$ , the lines  $L_{-1,0}$  and  $L_{-1,5}$  for  $i = 1, j = 3$ , and the vertical lines  $x = 1$  and  $x = 2$  for  $i = 2, j = 3$ ) as well as the lines  $p_i q_i$  for  $i = 2, 3$ . (The latter are horizontal Euclidean lines  $y = 3$  and  $y = -1$ .) However, the third line  $p_1 p'_1$  is not parallel to the other two so that Desargues' configuration does not close in  $\mathcal{A}_1$ .  $\square$

The lemma below will be used to prove that two distinct circles have at most two points in common.

**LEMMA 5.3** *The following polynomial in  $t$  always has exactly one real root if  $x \neq y$ :*

$$f_{x,y,z,w}(t) = 4(x^3 - y^3)t^3 + 3(x^2z - y^2w)t^2 + 2(xz^2 - yw^2)t + z^3 - w^3.$$

Before we give the proof we remark that during the proof we have to determine the solutions to a system of two equations of in degree 6 in two variables. By the well-known result on the unsolvability by radicals of polynomials of degree 5 and higher, we cannot expect to have a nice and elegant analytical solution for this. Our method consists of reducing the above system of equations to high degree equations in one variable. Only at this point do we rely on Maple to calculate the roots of these equations. As the explicit expressions are long, tedious and not enlightening at all, we have opted to leave them out. Alternately, we could have used the well-developed and accurate method of Gröbner bases to determine solutions to systems of polynomial equations; compare [21].

*Proof.* As is well known, see, for example, [1], a polynomial  $f(t) = at^3 + bt^2 + ct + d$  has exactly one (counted with multiplicity) real root if and only if the discriminant

$$\Delta = -4b^3d + b^2c^2 - 4ac^3 + 18abcd - 27a^2d^2$$

is negative. We investigate the discriminant of  $f$  where  $a = 4(x^3 - y^3)$ ,  $b = 3(x^2z - y^2w)$ ,  $c = 2(xz^2 - yw^2)$  and  $d = z^3 - w^3$ . If  $y = 0$ , then the discriminant becomes

$$-4x^6(-135z^3w^3 + 50z^6 + 108w^6)$$

and if  $w = 0$ , then it becomes

$$-4x^6(-135z^3w^3 + 50z^6 + 108w^6).$$

Hence, the discriminant is negative unless one of the other coordinates is also zero. It is easy to see that in all those cases the expression is negative if  $x \neq y$ . Suppose from now on that  $yw \neq 0$  and denote  $\frac{x}{y}$  by  $\alpha$  and  $\frac{z}{w}$  by  $\beta$ , respectively. Then

$$\Delta = -4y^6w^6E(\alpha, \beta)$$

where  $E(\alpha, \beta)$  is a polynomial in  $\alpha$  and  $\beta$ . Hence, it suffices to prove that  $E$  is always positive if  $x \neq y$ . As a polynomial,  $E(\alpha, \beta)$  is continuous and differentiable. An explicit calculation reveals that  $E$  is bounded below, hence it suffices to prove that the only critical point for  $E$  occurs at  $(\alpha, \beta) = (1, 1)$  and hence is an absolute minimum for  $E$ . The critical points are the solutions of the following system of equations

$$E_1 = \frac{\partial E}{\partial \alpha} = 0; E_2 = \frac{\partial E}{\partial \beta} = 0.$$

Furthermore, calculation yields that  $E_1 - E_2$ ,  $\alpha E_1 - \beta E_2$  and  $\alpha^2 E_1 - \beta^2 E_2$  all contain  $\alpha - \beta$  as a factor. If  $\alpha = \beta$  we obtain a high degree equation in  $\beta$ , yielding only  $(\alpha, \beta) = (1, 1)$  as critical point. Therefore, from now on, we assume  $\alpha \neq \beta$ , and factor  $\alpha - \beta$  out in the above expressions. This yields enough independent equations to obtain a high degree equation in  $\beta$ , which yields no further critical points.  $\square$

We say that two circles touch analytically at a point  $p = (x_p, y_p) \in \mathbb{R}^2$  if they pass through  $p$  and their describing functions have equal derivatives at  $x_p$ , where we consider one-sided derivatives in case  $x_p = 0$ . Given the form of the circles, the circles  $C_{a_1, b_1, c_1}$  and  $C_{a_2, b_2, c_2}$  touch analytically at  $p$  if and only if  $h_{a_1, b_1}(x_p) + c_1 = h_{a_2, b_2}(x_p) + c_2$  and  $h'_{a_1, b_1}(x_p) = h'_{a_2, b_2}(x_p)$ .

**PROPOSITION 5.4** *Two distinct circles of  $\mathcal{L}(q)$  have at most two points in common. Moreover, two circles in  $\mathcal{L}(q)$  touch analytically at a point  $p \in \mathbb{R}^2$  if and only if they are tangent to each other at  $p$ .*

*Proof.* Let  $C_{a_1, b_1, c_1}$  and  $C_{a_2, b_2, c_2}$  be two distinct circles. If  $a_1 = a_2$ , then the first statement follows from Propositions 5.1 and 5.2 and using automorphisms  $\gamma_{1, s, 0}$ .

We now assume that  $a_1 \neq a_2$ . For the  $x$ -coordinate of a point of intersection one has

$$h_{a_1, b_1}(x) - h_{a_2, b_2}(x) = c_2 - c_1, \quad x \geq 0 \quad \text{and} \quad h_{a_1, b_1}(x) - h_{a_2, b_2}(x) = \frac{c_2 - c_1}{q}, \quad x < 0. \quad (*)$$

We consider the left-hand side  $h(x) = h_{a_1, b_1}(x) - h_{a_2, b_2}(x)$  of these equations.  $h(x)$  is a quartic polynomial in  $x$  with  $h(0) = 0$ . Explicitly,

$$\begin{aligned} h(x) &= (a_1^3 - a_2^3)x^4 + (a_1^2 b_1 - a_2^2 b_2)x^3 + (a_1 b_1^2 - a_2 b_2^2)x^2 + (b_1^3 - b_2^3)x, \\ h'(x) &= 4(a_1^3 - a_2^3)x^3 + 3(a_1^2 b_1 - a_2^2 b_2)x^2 + 2(a_1 b_1^2 - a_2 b_2^2)x + (b_1^3 - b_2^3). \end{aligned}$$

We want to show that  $y = h(x)$  has at most two intersection points with any given horizontal line if  $a_1 \neq a_2$ . Then, because  $h(0) = 0$ ,  $h(x) = r$  has one positive and one negative solution in case  $r > 0$  and solutions have the same sign in case  $r < 0$ . But  $c_2 - c_1$  and  $\frac{c_2 - c_1}{q}$  have the same sign so that for  $c_2 > c_1$  we obtain one solution for each of the two equations (\*). For  $c_2 < c_1$  one of the equations in (\*) has no solution and the other at most 2, and for  $c_2 = c_1$  we obtain one or two solutions.

Therefore, it suffices to prove that  $h'(x)$  has exactly one real root in this case. This is achieved by the application of Lemma 5.3 since  $h'(x) = f_{a_1, a_2, b_1, b_2}(x)$ .

In case  $h(x_0) = r$  and  $h'(x_0) = 0$ , the equation  $h(x) = r$  has only one solution  $x_0$ . It follows that the circles are tangent at  $(x_0, y_0)$ . Conversely, if two circles are tangent to one another at a point  $(x_0, y_0)$ , then  $h'(x_0)$  has to be zero, otherwise we surely get a second point of intersection since  $h$  is a polynomial of degree 4.  $\square$

**REMARK 5.5** In some cases we can prove Proposition 5.4 in a shorter way. Recall that  $h_{a,b}$  is strictly convex for  $a > 0$ , strictly concave for  $a < 0$  and linear for  $a = 0$ . Hence, if  $a_1 \geq 0 \geq a_2$ , the function  $h$  itself is strictly convex, and an equation  $h(x) = r$  has at most 2 solutions  $x$ . As seen in the proof of Proposition 5.4 this property suffices to prove the assertion of the Proposition.

A similar argument applies when  $a_1 < 0 < a_2$ , or we can use the automorphism  $\gamma_{1,-1,0}$  to reduce this situation to the case considered above. Using  $\gamma_{1,-1,0}$ , if necessary, it suffices to consider the case where  $a_1 > a_2 > 0$ . In this case we really need the approach used in Proposition 5.4.

**PROPOSITION 5.6** *Three mutually non-parallel points can be uniquely joined by a circle in  $\mathcal{L}(q)$ .*

*Proof.* Let  $(x_i, y_i)$ ,  $i = 1, 2, 3$ , be three mutually non-parallel points. If any one of the  $x_i$ s is  $\infty$ , the result follows from Propositions 5.1 and 5.2. So we can assume that all  $x_i$  are in  $\mathbb{R}$  and, without loss of generality, furthermore that  $x_1 > x_2 > x_3$ .

We consider the generic case where  $x_3 < 0 \leq x_2$ . In this case we have to find  $a, b, c \in \mathbb{R}$  such that

$$\begin{aligned} y_1 &= h_{a,b}(x_1) + c \\ y_2 &= h_{a,b}(x_2) + c \\ y_3 &= qh_{a,b}(x_3) + c \end{aligned}$$

Subtracting the first equation from the other two yields

$$\begin{aligned} y_2 - y_1 &= h_{a,b}(x_2) - h_{a,b}(x_1) \\ y_3 - y_1 &= qh_{a,b}(x_3) - h_{a,b}(x_1) \end{aligned}$$

which does not involve  $c$ .

We first consider the case that the three modified points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(qx_3, y_3)$  are on a Euclidean line. Then  $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{qx_3 - x_1}$  and

$$a = 0, \quad b = \sqrt[3]{\frac{y_2 - y_1}{x_2 - x_1}}, \quad c = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

yields a circle containing the three original points.

If the three modified points are not on a Euclidean line, then  $(y_2 - y_1)(qx_3 - x_1) - (y_3 - y_1)(x_2 - x_1) \neq 0$  and  $a$  cannot be 0. We now write  $h_{a,b}(x)$  as  $a^3 h_{1,t}(x)$  where  $t = b/a$ . From the above system of two equations for  $a$  and  $b$  we then obtain the single equation

$$(y_2 - y_1)(qh_{1,t}(x_3) - h_{1,t}(x_1)) - (y_3 - y_1)(h_{1,t}(x_2) - h_{1,t}(x_1)) = 0$$

for  $t$ . Expanding the product yields a cubic polynomial in  $t$  with  $t^3$  having coefficient  $(y_2 - y_1)(qx_3 - x_1) - (y_3 - y_1)(x_2 - x_1) \neq 0$ . Hence this polynomial has at least one real root  $t_0$ . From Proposition 5.4 it follows that not both of  $h_{1,t_0}(x_2) - h_{1,t_0}(x_1)$  and  $qh_{1,t_0}(x_3) - h_{1,t_0}(x_1)$  can be zero. (Otherwise the circle  $C_{1,t_0,0}$  has three points in common with  $C_{0,0,0}$ .) Hence we can find  $a$  as

$$a = \sqrt[3]{\frac{y_2 - y_1}{h_{1,t_0}(x_2) - h_{1,t_0}(x_1)}} \quad \text{or} \quad a = \sqrt[3]{\frac{y_3 - y_1}{qh_{1,t_0}(x_3) - h_{1,t_0}(x_1)}}.$$

From there we obtain  $b = at_0$  and then  $c = y_1 - h_{a,b}(x_1)$ . This shows that the three points are on a circle. Furthermore, in any case, a joining circle must be unique by Proposition 5.4.

If we assume  $x_3 \geq 0$ , we can put  $q = 1$  in the above computations and obtain in exactly the same way a unique joining circle. In the two remaining cases  $x_1 < 0$  and  $x_2 < 0 \leq x_1$  we use the isomorphism  $\sigma$  to reduce them to the previous cases in  $\mathcal{L}(1/q)$ .  $\square$

**PROPOSITION 5.7** *Consider a circle  $\tilde{C}$  and two non-parallel points  $p_1 = (x_1, y_1) \notin \tilde{C}$  and  $p_2 = (x_2, y_2) \in \tilde{C}$ . Then there exists exactly one circle  $C_{a,b,c}$  through  $p_1$  which intersects  $\tilde{C}$  exactly in  $p_2$ .*

*Proof.* If  $p_2$  is on the parallel class at infinity  $\Pi_\infty$ , the result follows from Propositions 5.1 and 5.2. If  $p_2$  is not on  $\Pi_\infty$ , then it suffices to prove existence of a touching circle. Indeed, suppose there were two circles  $C$  and  $C'$  fulfilling the requirements. Since analytic and geometric touching are the same by Proposition 5.4 and because both circles touch  $\tilde{C}$  analytically, they also touch each other analytically. Hence, by Proposition 5.4, they

must touch geometrically. But they have  $p_1$  and  $p_2$  in common by assumption, which yields a contradiction.

If  $p_1 \in \Pi_\infty$ , then  $a = \tilde{a}$  is known, where  $\tilde{C} = C_{\tilde{a}, \tilde{b}, \tilde{c}}$ , and expressing that  $C = C_{a,b,c}$  and  $\tilde{C}$  touch at  $p_2$  yields a cubic equation in  $b$ :

$$4x_2^3a^3 + 3x_2^2a^2b + 2x_2ab^2 + b^3 = h'_{\tilde{a}, \tilde{b}}(x_2) = y'_2. \quad (1)$$

This equation has a real root and the remaining parameter  $c$  can be found from the equation expressing that  $p_2$  lies on  $C$  (put  $q = 1$  if  $x_2 \geq 0$ ):

$$y_2 = q(x_2^4a^3 + x_2^3a^2b + x_2^2ab^2 + x_2b^3) + c.$$

Now we deal with the case where  $p_1, p_2 \in \mathbb{R}^2$ . Without loss of generality we may assume that  $x_1 \geq 0$  and  $x_2 < 0$ . The other cases follow by either putting  $q = 1$  in the calculations or by isomorphism.

Since  $p_1$  and  $p_2$  lie on  $C$ , we have that

$$y_1 = x_1^4a^3 + x_1^3a^2b + x_1^2ab^2 + x_1b^3 + c, \quad (2)$$

$$y_2 = q(x_2^4a^3 + x_2^3a^2b + x_2^2ab^2 + x_2b^3) + c. \quad (3)$$

Subtracting (2) from (3) yields

$$y_2 - y_1 = (qx_2^4 - x_1^4)a^3 + (qx_2^3 - x_1^3)a^2b + (qx_2^2 - x_1^2)ab^2 + (qx_2 - x_1)b^3. \quad (4)$$

Expressing that  $C$  and  $\tilde{C}$  touch analytically at  $p_2$  yields again equation (1) after dividing both sides by  $q$ .

In case  $\frac{y_2 - y_1}{qx_2 - x_1} = y'_2$ , we can take  $a = 0$ ,  $b = \sqrt[3]{y'_2}$ ,  $c = y_1 - x_1y'_2$ .

In case  $\frac{y_2 - y_1}{qx_2 - x_1} \neq y'_2$ , we must have  $a \neq 0$ . We then write  $b = ta$  and (3) and (4)

become

$$((qx_2^4 - x_1^4) + (qx_2^3 - x_1^3)t + (qx_2^2 - x_1^2)t^2 + (qx_2 - x_1)t^3)a^3 = y_2 - y_1, \quad (5)$$

$$(4x_2^3 + 3x_2^2t + 2x_2t^2 + t^3)a^3 = y'_2. \quad (6)$$

Eliminating  $a$  from equations (5) and (6) yields a cubic equation in  $t$ , which has a real root  $t_0$ . Note that the polynomials  $4x_2^3 + 3x_2^2t + 2x_2t^2 + t^3$  and  $(qx_2^4 - x_1^4) + (qx_2^3 - x_1^3)t + (qx_2^2 - x_1^2)t^2 + (qx_2 - x_1)t^3$  have no roots in common. Indeed, combining the two equations yields a quadratic equation which after putting  $x_1 = \lambda x_2$  has discriminant

$$-x_2^6(3\lambda^6 - 8\lambda^5 + (6 + 4q)\lambda^4 + (-16 - 4q)\lambda^3 + (23 + 12q)\lambda^2 - 28\lambda q + 8q^2),$$

an expression which is always negative since  $\lambda < 0$ . This implies we can solve for  $a$  from either (5) or (6). This also gives us  $b$ , and finally we find  $c$  from either (2) or (3).  $\square$

**THEOREM 5.8**  $\mathcal{L}(q)$  is a flat Laguerre plane of group dimension 3. Furthermore,  $\mathcal{L}(q)$  is of Kleininghoffer type III.B.1 for  $q \neq 1$  and of type III.B.3 for  $q = 1$ .

*Proof.* The previous propositions prove that  $\mathcal{L}(q)$  is a flat Laguerre plane. It is readily verified that  $\{\gamma_{1,s,(1-s)c} \mid s \in \mathbb{R}, s \neq 0\}$  is a linearly transitive group of  $C_{0,0,c}$ -homologies. Hence, the set  $\mathcal{Z}$  of all circles for which the automorphism group  $\Gamma(q)$  of  $\mathcal{L}(q)$  is linearly transitive with respect to  $C$ -homologies contains circles as in type III.

We first note that every automorphism of  $\mathcal{L}(q)$  fixes the point  $(\infty, 0)$ . Assume otherwise, that is, that there is an automorphism  $\gamma$  such that  $\gamma((\infty, 0)) = p \neq (\infty, 0)$ . Then  $\gamma(\{C_{0,0,t} \mid t \in \mathbb{R}\})$  is the bundle  $B(p, \gamma(C_{0,0,0}))$  and this bundle must also be contained in  $\mathcal{Z}$ . From the list of possible types with respect to Laguerre homologies, compare [14], Section 3, and because type VI is not possible in flat Laguerre planes by [14], Corollary 3.3, we obtain that  $\mathcal{L}(q)$  must then be of type VII. However, such a flat Laguerre plane is ovoidal by [14], Corollary 3.2. This contradicts the fact that  $\mathcal{A}_{(\infty,1)}$  is non-Desarguesian by Lemma 5.2. (Each derived affine plane of an ovoidal Laguerre plane is Desarguesian.)

Furthermore, because a non-identity Laguerre homology about a circle not passing through  $(\infty, 0)$  moves  $(\infty, 0)$ , any circle  $C$  for a  $C$ -homology must pass through  $(\infty, 0)$ . The same argument on circles in  $\mathcal{Z}$  as above yields that if there is a linearly transitive group of Laguerre homologies about a circle  $C$ , then this circle must be in the tangent bundle  $\{C_{0,0,c} \mid c \in \mathbb{R}\}$ . This shows that  $\mathcal{L}(q)$  must be of type III with respect to Laguerre homologies.

Since  $(\infty, 0)$  is fixed, every automorphism  $\alpha$  of  $\mathcal{L}(q)$  induces a collineation of the derived affine plane  $\mathcal{A}_{(\infty,0)}$ . But  $\mathcal{A}_{(\infty,0)}$  is Desarguesian so that in  $\mathcal{A}_{(\infty,0)}$  we have  $\alpha(x, y) = (rx + u, sy + vx + t)$  for some  $r, s, t, u, v \in \mathbb{R}$ ,  $r, s \neq 0$ . Applying suitable automorphisms of  $\mathcal{L}(q)$  we may assume that  $r = \pm 1$ ,  $s = 1$  and  $t = 0$ . Now in case  $r = 1$

$$\begin{aligned} \alpha(C_{1,0,0}) &= \{\alpha((x, x^4) \mid x \in \mathbb{R}, x \geq 0)\} \cup \{\alpha((x, qx^4) \mid x \in \mathbb{R}, x < 0)\} \cup \{\alpha((\infty, 1))\} \\ &= \{(x + u, x^4 + vx) \mid x \in \mathbb{R}, x \geq 0\} \cup \{(x + u, qx^4 + vx) \mid x \in \mathbb{R}, x < 0\} \\ &\quad \cup \{\alpha((\infty, 1))\} \\ &= \{(z, (z - u)^4 + v(z - u)) \mid z \geq u\} \cup \{(z, q(z - u)^4 + v(z - u)) \mid z < u\} \\ &\quad \cup \{\alpha((\infty, 1))\} \\ &= \{(z, z^4 - 4uz^3 + 6u^2z^2 + (v - 4u^3)z + u^4 - vu) \mid z \geq u\} \\ &\quad \cup \{(z, qz^4 - 4quz^3 + 6qu^2z^2 + (v - 4qu^3)z + qu^4 - vu) \mid z < u\} \\ &\quad \cup \{\alpha((\infty, 1))\} \end{aligned}$$

To match this set with a circle  $C_{a,b,c}$  we must have  $a = 1$ ,  $b = -4u$ ,  $b^2 = 6u^2$ ,  $b^3 = v - 4u^3$  and  $c = vu$ . The second and third of these equations yield  $u = 0 = b$ , and then the fourth equation gives us  $v = 0$ . Hence  $\alpha = \text{id}$ . A similar computation in case  $r = -1$  leads to  $\alpha = \sigma$ , and this  $\alpha$  is an automorphism if and only if  $q = 1$ .

This shows that  $\Gamma(q)$  equals  $G = \{\gamma_{r,s,t} \mid r, s, t \in \mathbb{R}, s \neq 0, r > 0\}$  in case  $q \neq 1$  and  $\langle G, \sigma \rangle = G \cup \sigma G$  in case  $q = 1$ . In any case  $\Gamma(q)$  is 3-dimensional. Furthermore,  $\Gamma(q)$  fixes the parallel class  $\Pi_0$  and is transitive on  $\Pi_0$ .

Since  $\{\gamma_{1,1,t} \mid t \in \mathbb{R}\}$  is a linearly transitive group of  $\Pi_\infty$ -translations and because a non-identity  $(|p|, B(p, C))$ -translation moves at least one of  $\Pi_\infty$  or  $\Pi_0$ , we see that  $\mathcal{L}(q)$  must be of type B with respect to Laguerre translations. Similarly, a non-identity  $(p, q)$ -homothety with centres  $p, q$  not on  $\Pi_\infty \cup \Pi_0$  moves at least one of  $\Pi_\infty$  or  $\Pi_0$ , and a non-identity  $(p, q)$ -homothety with  $p \in \Pi_\infty \setminus \{(\infty, 0)\}$  and  $q \in \Pi_0$  moves  $(\infty, 0)$ . Hence, only  $\{p, q\} = \{(\infty, 0), q\}$  with  $q \in \Pi_0$  can occur for linearly transitive groups of  $\{p, q\}$ -homotheties. But  $\{\gamma_{r,r,(1-r)c} \mid r \in \mathbb{R}, r \neq 0\}$  is a linearly transitive group of  $((\infty, 0), (0, c))$ -homotheties in case  $q = 1$  so that  $\mathcal{L}(1)$  is of type 3. For  $q \neq 1$  however,  $\sigma$  is not an automorphism of  $\mathcal{L}(q)$  and the group of all  $((\infty, 0), (0, 0))$ -homotheties is not linearly transitive. We therefore obtain Kleinewillinghöfer type 1 in this case.  $\square$

**THEOREM 5.9** *Two flat Laguerre planes  $\mathcal{L}(q)$  and  $\mathcal{L}(q')$  are isomorphic if and only if  $q' = q$  or  $q'q = 1$ . In particular, each Laguerre plane  $\mathcal{L}(q)$  is isomorphic to precisely one plane with  $q \geq 1$ . Furthermore, each isomorphism from  $\mathcal{L}(q)$  to  $\mathcal{L}(q')$  is a composition of automorphisms of  $\mathcal{L}(q)$  and the isomorphism  $\sigma$ .*

*Proof.* Let  $\varphi$  be an isomorphism from  $\mathcal{L}(q)$  to  $\mathcal{L}(q')$ . Since both Laguerre planes are of type III.B, the point  $(\infty, 0)$  in  $\mathcal{L}(q)$  must be taken to the point  $(\infty, 0)$  in  $\mathcal{L}(q')$ . Hence  $\varphi$  induces an isomorphism between the derived affine planes  $\mathcal{A}_{(\infty,0)}$  and  $\mathcal{A}'_{(\infty,0)}$ , and thus a collineation of the real Desarguesian plane. As in the proof of Theorem 5.8 we see that  $\varphi$  is a composition of an automorphism of  $\mathcal{L}(q)$  followed perhaps by  $\sigma$ . In the former case  $q' = q$  and in the latter case  $q' = 1/q$ .

The remaining statements of the theorem now readily follow.  $\square$

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