

Title: Characterizations of finite classical polar spaces by intersection numbers with hyperplanes and spaces of codimension 2

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Characterizations of finite classical polar spaces by intersection numbers with hyperplanes and spaces of codimension 2

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Abstract

In this article we show that non-singular quadrics and non-singular Hermitian varieties are completely characterized by their intersection numbers with respect to hyperplanes and spaces of codimension 2. This strongly generalizes a result by Ferri and Tallini [4] and also provides necessary and sufficient conditions for quasi-quadrics (respectively their Hermitian analogues) to be non-singular quadrics (respectively Hermitian varieties).

1 Introduction

When Segre [12] proved his celebrated characterization of conics (“every set of $q+1$ points in $\text{PG}(2, q)$, q odd, no three of which are collinear, is a conic”), he did more than proving a beautiful and interesting theorem; he in fact provided the starting point of a new direction in combinatorial geometry. In this branch of combinatorics the idea is to provide purely combinatorial characterizations of objects classically defined in an algebraic way. This article wants to contribute to this theory by proving strong characterizations of classical finite polar spaces.

1.1 Polar spaces

Though we suppose the reader to be familiar with polar spaces, that is, via polarities, quadrics and Hermitian varieties in $\text{PG}(n, q)$, we will quickly overview notation and terminology and recall some results that will be useful throughout this paper.

A finite classical non-singular polar space is one of the following:

- a hyperbolic quadric in $\text{PG}(2n + 1, q)$, denoted by $Q^+(2n + 1, q)$;
- a parabolic quadric in $\text{PG}(2n, q)$, denoted by $Q(2n, q)$;
- an elliptic quadric in $\text{PG}(2n + 1, q)$, denoted by $Q^-(2n + 1, q)$;
- a Hermitian variety in $\text{PG}(n, q^2)$, denoted by $H(n, q^2)$;
- a symplectic polar space in $\text{PG}(2n + 1, q)$, denoted by $W_{2n+1}(q)$.

Remark. Throughout this article we will use the above notations for non-singular polar spaces. A cone with vertex a point p or a line L over a non-singular polar space, e.g. over a $Q(2n, q)$ will be denoted as $pQ(2n, q)$, respectively $LQ(2n, q)$ (a small letter will always indicate the vertex of the cone is a point, while a capital letter will indicate the vertex is a line). With these conventions, the notation will always immediately tell whether the considered polar space is singular or non-singular. Only in the statements of our lemmas and theorems we will explicitly mention the (non-)singular character of the considered polar spaces.

1.2 Characterizations of polar spaces

A *thick partial linear space of order (s, t)* , is a point-line geometry such that each line contains $s + 1 > 2$ points, such that each point is contained in $t + 1 > 2$ lines, and such that two distinct lines intersect in at most one point.

A thick partial linear space S is said to be a *Shult space* if for every antiflag (p, L) of S either exactly 1 or all points of L are collinear with p .

The following very nice theorem is the result of work done by Veldkamp [17], Tits [16], Buekenhout and Shult [2].

Theorem 1.1 *Suppose that S is a Shult space such that no point of S is collinear with all other points of S , and such that there exists an antiflag (p, L) with the property that p is collinear with all points of L . Then S is isomorphic to the point-line geometry of a finite classical non-singular polar space.*

If a Shult space is fully embedded in a projective space then the following theorem follows from Buekenhout and Lefèvre [1], and Lefèvre-Percsy [8, 9].

Theorem 1.2 *Suppose S is a Shult space such that no point of S is collinear with all other points of S . If S is fully embedded in a projective space, then S consists of the points and lines of a finite classical polar space. (Here fully embedded means that the set of lines of S is a subset of the set of lines of the projective space and that the point set of S is the set of all points contained in these lines.)*

The following characterizations of quadrics and Hermitian varieties, which can be found in Hirschfeld and Thas [7], will be of great importance in the proof of our main theorem. These theorems also provide nice examples of Segre-type theorems.

Definition A point set K in $PG(n, q)$ is said to be of type (r_1, r_2, \dots, r_s) if $|L \cap K| \in \{r_1, r_2, \dots, r_s\}$ for all lines L of $PG(n, q)$. A point $p \in K$ is called *singular* with respect to K if all lines through p intersect K either in 1 or in $q+1$ points. If a set K contains a singular point, then K is called *singular*.

The theorem below is an amalgamation of results by Tallini-Scafati [15], Hirschfeld and Thas [6] and Glynn [5].

Theorem 1.3 *Let K be a non-singular point set of type $(1, r, q^2 + 1)$ in $PG(n, q^2)$, $n \geq 4$ and $q > 2$, satisfying the following properties :*

- $3 \leq r \leq q^2 - 1$;
- *there does not exist a plane π such that every line of π intersects $\pi \cap K$ in r or $q^2 + 1$ points;*

Then the set K is the point set of a non-singular Hermitian variety $H(n, q^2)$.

The result below was obtained by Tallini in [13] and [14].

Theorem 1.4 *In $PG(n, q)$ with $n \geq 4$ and $q > 2$, let K be a non-singular point set of type $(0, 1, 2, q + 1)$.*

If $\frac{q^{n+1}-1}{q-1} > |K| \geq \frac{q^n-1}{q-1}$ then one of the following holds:

- (i) $|K| = \frac{q^n-1}{q-1}$, n is even and K is the point set of a $Q(n, q)$.
- (ii) $|K| = \frac{q^n-1}{q-1} + q^{\frac{n-1}{2}}$, n is odd and K is the point set of a $Q^+(n, q)$.
- (iii) $|K| = \frac{q^n-1}{q-1} + 1$, q is even, and $K = \Pi_t K' \cup \{N\}$ with Π_t some $PG(t, q) \subset PG(n, q)$ and with K' (the point set of) a $Q(n-t-$

1, q) in some $(n - t - 1)$ -dimensional subspace of $\text{PG}(n, q)$ skew from $\text{PG}(t, q)$ (hence $n - t - 1$ is even) or with K' a $(q + 1)$ -arc in plane skew from $\text{PG}(t, q)$ if $t = n - 3$. In each case N is the nucleus of K' .

These two theorems provide characterizations of certain polar spaces by their line intersections. It is a natural question to ask whether polar spaces can also be characterized by their intersections with respect to other subspaces. A first result in this direction was given by the following theorem of Ferri-Tallini [4].

Theorem 1.5 *If a set K of points in $\text{PG}(n, q)$, $n \geq 4$, with $|K| \geq q^3 + q^2 + q + 1$, is such that it has as intersection numbers with planes 1, a , b , where $a \leq b$ and $b \geq 2q + 1$, and as intersection numbers with solids c , $c + q$, $c + 2q$, and such that solids intersecting K in c and $c + q$ points exist, then K is the point set of a $Q(4, q)$.*

This theorem is a clear motivation to look at the following questions:

Is it possible to characterize finite classical polar spaces by their intersection numbers with respect to planes and solids, respectively by their intersections with respect to hyperplanes and subspaces of codimension 2?

(Note that these questions of course do not make any sense for the polar space $W_{2n+1}(q)$, as this polar space comprises all points of its ambient projective space.) The first question was answered affirmatively in Schillewaert [10], and Schillewaert and Thas [11]; the second question will be the subject of this paper. Though it is possible to characterize certain polar spaces only by their line intersections, the existence (in abundance) of quasi-quadrics and quasi-Hermitian varieties shows that it is not possible to characterize them merely by their intersections with respect to hyperplanes. Here we define a quasi-quadric, respectively quasi-Hermitian variety, to be a subset of the set of points of a projective space having the same intersection numbers with respect to hyperplanes as a non-singular quadric, respectively a non-singular Hermitian variety. In the parabolic case this is a slight deviation of the standard definition as given in [3]. The concept of a quasi-Hermitian variety in fact does not appear in the literature, but in unpublished work the second author showed that examples can be constructed with the same techniques as used to construct quasi-quadrics. For an overview on quasi-quadrics we refer to [3].

Hence the following Segre-type characterization of polar spaces, which is the main theorem of this paper, also provides necessary and sufficient conditions for a quasi-quadric or a quasi-Hermitian variety to be a non-singular quadric or Hermitian variety.

Theorem 1.6 *If a point set K in $\text{PG}(n, q)$, $n \geq 4$, $q > 2$, has the same intersection numbers with respect to hyperplanes and subspaces of codimension 2 as a polar space $P \in \{H(n, q), Q^+(n, q), Q^-(n, q), Q(n, q)\}$, then K is the point set of a non-singular polar space P .*

Remark For $n = 3$ the conclusion of the above theorem remains true in the Hermitian and hyperbolic case. This is easily seen using Theorem 1.2. In the elliptic case the conclusion only remains true if q is odd, as every non-classical ovoid provides a counter example. If $n = 4$ the conclusion of the theorem is true for all q . In the parabolic case this is a corollary of Theorem 1.5, see Schillewaert [10], and in the Hermitian case this result was obtained in Schillewaert and Thas [11].

In the next section we will handle the four different types of polar spaces one by one. The basic idea is to study certain structures in the dual projective space. However the elliptic and parabolic case will turn out to be harder than the other two cases, and especially the parabolic case will be more complex and interesting (this is basically due to the fact that there are more intersections with respect to hyperplanes).

2 Proof of the Theorem 1.6

From here on we always assume that we work in a projective space of dimension at least four and that $q > 2$.

2.1 Hermitian varieties

Let us first recall the intersections of an $H(n, q^2)$ in $\text{PG}(n, q^2)$ with hyperplanes and subspaces of codimension 2. A hyperplane intersects an $H(n, q^2)$ either in an $H(n-1, q^2)$ or in a cone $pH(n-2, q^2)$. A subspace of codimension 2 intersects an $H(n, q^2)$ either in an $H(n-2, q^2)$, in a cone $pH(n-3, q^2)$ or in a cone $LH(n-4, q^2)$. Hence the intersection numbers with hyperplanes are

$$H_1 = \frac{(q^n - (-1)^n)(q^{n-1} + (-1)^n)}{q^2 - 1}, H_2 = 1 + \frac{q^2(q^{n-1} + (-1)^n)(q^{n-2} - (-1)^n)}{q^2 - 1}.$$

and the intersection numbers with subspaces of codimension 2 are

$$C_1 = \frac{(q^{n-1} + (-1)^n)(q^{n-2} - (-1)^n)}{q^2 - 1}, C_2 = 1 + \frac{q^2(q^{n-2} - (-1)^n)(q^{n-3} + (-1)^n)}{q^2 - 1},$$

$$C_3 = 1 + q^2 + \frac{q^4(q^{n-3} + (-1)^n)(q^{n-4} - (-1)^n)}{q^2 - 1}.$$

From now on, let K be a point set in $\text{PG}(n, q^2)$ having the above intersection numbers with respect to hyperplanes and subspaces of codimension 2. We want to prove that K is the point set of an $H(n, q^2)$. We will call subspaces intersecting K in a given number m of points, *subspaces of type m* . For obvious reasons a hyperplane intersecting K in H_2 points will also be called a *tangent hyperplane*.

Lemma 2.1 *The set K contains $|H(n, q^2)|$ points. There are $|H(n, q^2)|$ tangent hyperplanes.*

Proof We count in two ways the pairs (p, α) where p is a point of K and α a hyperplane such that $p \in \alpha$, respectively the triples (p, r, α) where $p \neq r$ are points of K and α a hyperplane such that $p, r \in \alpha$. Denote by h_1 the number of hyperplanes intersecting K in H_1 points and by x the size of K . We obtain:

$$x \frac{q^{2n} - 1}{q^2 - 1} = h_1 H_1 + \left(\frac{q^{2n+2} - 1}{q^2 - 1} - h_1 \right) H_2, \quad (1)$$

and

$$x(x-1) \frac{q^{2n-2} - 1}{q^2 - 1} = h_1 H_1 (H_1 - 1) + \left(\frac{q^{2n+2} - 1}{q^2 - 1} - h_1 \right) H_2 (H_2 - 1). \quad (2)$$

Solving the first equation for h_1 and substituting this value in the second equation yields a quadratic equation in x . The solutions are $x_1 = |H(n, q^2)|$ and x_2 , which is a tedious expression in q , however easily computed with any computer algebra package.

We show the latter is impossible. So suppose by way of contradiction that the set K contains x_2 points. Consider a subspace Π of codimension 2 intersecting the set K in C_i points. Denote by k_i the number of non-tangent hyperplanes containing Π . We obtain the following equation.

$$k_i(H_1 - C_i) + (q^2 + 1 - k_i)(H_2 - C_i) + C_i = x_2.$$

Solving this equation in k_i for $i = 1, 2, 3$ yields respectively

$$k_1 = \frac{q^n - (-1)^n(q^2 - q + 1)}{q^{n-1} - (-1)^n}; \quad k_2 = \frac{2q^n - (-1)^n(q^2 + 1)}{q^{n-1} - (-1)^n};$$

$$k_3 = -\frac{q^{n+1} - 2q^n + (-1)^n}{q^{n-1} - (-1)^n}.$$

These are not natural numbers if $n > 2$, proving x_2 cannot occur.

The second assertion follows by substituting $x = |H(n, q^2)|$ in Equation 1. ■

Remark. Notice that for $n = 2$, we would obtain natural numbers and in that case we have $x_2 = q^2 + q + 1$, that is, exactly the number of points of a Baer subplane, which was to be expected.

Lemma 2.2 *Through a space of codimension 2 of type C_1, C_2, C_3 there pass respectively $T_1 = q + 1, T_2 = 1, T_3 = q^2 + 1$ tangent hyperplanes.*

Proof Let Π be a codimension 2-space intersecting K in C_i points and let T_i denote the number of tangent hyperplanes containing Π . We obtain the following equation:

$$C_i + T_i(H_2 - C_i) + (q^2 + 1 - T_i)(H_1 - C_i) = |K|.$$

Solving the equation in T_i for $i = 1, 2, 3$ yields the result. \blacksquare

Lemma 2.3 *Each tangent hyperplane contains A_i subspaces of codimension 2 intersecting K in C_i points, where*

$$A_1 = q^{2n-2}, \quad A_2 = \frac{q^{n-2}(q^{n-1} + (-1)^n)}{q + 1},$$

$$A_3 = \frac{(q^{n-1} + (-1)^n)(q^{n-2} - (-1)^n)}{q^2 - 1}.$$

Proof Consider any tangent hyperplane Π . Denote by A_i the number of codimension 2 subspaces of type C_i contained in Π . Then

$$\sum_i A_i = \frac{q^{2n} - 1}{q^2 - 1}.$$

We count in two ways the pairs (p, Δ) , $p \in \Delta \subset \Pi$, with p a point of K and Δ a subspace of codimension 2. We obtain

$$\sum_i A_i C_i = H_2 \frac{q^{2n-2} - 1}{q^2 - 1}.$$

Next we count in two ways the triples (p, r, Δ) , with $p, r \in \Delta \subset \Pi$, with $p \neq r$ points of K and Δ a subspace of codimension 2. We obtain

$$\sum_i A_i C_i (C_i - 1) = H_2 (H_2 - 1) \frac{q^{2n-4} - 1}{q^2 - 1}.$$

The obtained system of three linear equations in A_1, A_2 and A_3 can easily be solved, yielding the desired result. \blacksquare

Lemma 2.4 *Each point of K is contained in H_2 tangent hyperplanes, while each point not in K is contained in H_1 tangent hyperplanes.*

Proof We need to show that each point of K is contained in H_2 tangent hyperplanes. Set $K = \{p_1, \dots, p_{|H(n, q^2)|}\}$. Let a_i denote the number of tangent hyperplanes containing the point p_i . Counting pairs (p, τ) , $p \in K$, $p \in \tau$, τ a tangent hyperplane we obtain:

$$\sum_i a_i = |H(n, q^2)|H_2. \quad (3)$$

Next we count triples (p, τ_1, τ_2) , $p \in K$, $p \in \tau_i$, τ_i a tangent hyperplane, $i = 1, 2$, $\tau_1 \neq \tau_2$. This yields the following equation.

$$\sum_i a_i(a_i - 1) = |H(n, q^2)| \left(\sum_j A_j(T_j - 1)C_j \right), \quad (4)$$

where as before T_i denotes the number of tangent hyperplanes containing a fixed codimension 2-space Π of type C_i and A_i is the number of codimension 2 subspaces of type C_i in a tangent hyperplane.

From Equations (3) and (4) we can compute

$$|H(n, q^2)| \sum_i a_i^2 - \left(\sum_i a_i \right)^2 = 0,$$

from which we deduce, using the variance trick, that a_i is a constant equal to H_2 .

The second assertion is proved in a similar way. ■

Denote by \mathcal{H} the set of tangent hyperplanes of K . Let $\delta : \text{PG}(n, q) \rightarrow \text{PG}(n, q)^D$ be any fixed chosen duality of $\text{PG}(n, q)$.

Lemma 2.5 *The point set $K' := \mathcal{H}^\delta$ is a $(1, q + 1, q^2 + 1)$ -set in $\text{PG}(n, q)^D$.*

Proof As by Lemma 2.2 a codimension 2 subspace is contained in either 1 , $q + 1$ or $q^2 + 1$ tangent hyperplanes, applying δ immediately yields the result. ■

Lemma 2.6 *The set K' is the point set of a non-singular Hermitian variety $H(n, q^2)$ in $\text{PG}(n, q)^D$.*

Proof We will check the conditions of Theorem 1.3. Since every tangent hyperplane contains subspaces of type C_1 (by Lemma 2.3), we see that every point of K' is contained in lines intersecting K' in exactly $q + 1$ points. Hence K' is non-singular. As $q > 2$ also

the first condition is satisfied. Now assume there would be a plane π intersecting K' in a point set such that every line of π would intersect $K' \cap \pi$ in $q+1$ or q^2+1 points. Then $K' \cap \pi$ would be the complement of a maximal $(q^2 - q)$ -arc in π , implying that $q^2 - q$ divides q^2 , a contradiction since $q > 2$. Consequently also the second condition of theorem 1.3 is satisfied and K' is the point set of a non-singular Hermitian variety in $\text{PG}(n, q^2)$. ■

Theorem 2.7 *The set K is the point set of a non-singular Hermitian variety $H(n, q^2)$.*

Proof Clearly K' is a point set satisfying the same conditions on intersections with hyperplanes and subspaces of codimension 2 as K in $\text{PG}(n, q^2)$. Since the tangent hyperplanes at K' are exactly those hyperplanes containing H_2 points of K' , we find that if we apply the duality δ^{-1} to $\text{PG}(n, q)^D$, the tangent hyperplanes of K' are mapped bijectively to the points of K (see Lemma 2.4). Hence, we can now apply Lemma 2.6 with K' in $\text{PG}(n, q^2)^D$ replaced by K in $\text{PG}(n, q^2)$. We conclude that K is the point set of an $H(n, q^2)$. ■

2.2 Hyperbolic quadrics

First of all let us recall what the intersections of a $Q^+(2n+1, q)$ with hyperplanes and spaces of codimension 2 look like. A hyperplane can intersect a $Q^+(2n+1, q)$ in a $Q(2n, q)$ or in a cone $pQ^+(2n-1, q)$. A space of codimension 2 can intersect a $Q^+(2n+1, q)$ either in a $Q^-(2n-1, q)$, in a cone $pQ(2n-2, q)$, in a $Q^+(2n-1, q)$ or in a cone $LQ^+(2n-3, q)$.

So the intersection numbers with hyperplanes are

$$H_1 = \frac{q^{2n} - 1}{q - 1}, \quad H_2 = 1 + q \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1}.$$

The intersection numbers with spaces of codimension 2 are

$$C_1 = \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1}, \quad C_2 = 1 + \frac{q(q^{2n-2} - 1)}{q - 1},$$

$$C_3 = \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1}, \quad C_4 = 1 + q + \frac{q^2(q^{n-1} - 1)(q^{n-2} + 1)}{q - 1}.$$

From now on let K be a set of points in $\text{PG}(2n+1, q)$ having the same intersection numbers with hyperplanes and codimension 2-spaces as a $Q^+(2n+1, q)$. We want to prove that K is the point set of a $Q^+(2n+1, q)$. We will call subspaces intersecting K in a

given number m of points, *subspaces of type m* . For obvious reasons a hyperplane intersecting K in H_2 points will also be called a *tangent hyperplane*.

Lemma 2.8 *The set K has size $|Q^+(2n+1, q)|$. Furthermore, there are $|Q^+(2n+1, q)|$ tangent hyperplanes.*

Proof We count in two ways the pairs (p, α) where p is a point of K and α a hyperplane such that $p \in \alpha$, respectively the triples (p, r, α) where $p \neq r$ are points of K and α a hyperplane such that $p, r \in \alpha$. Call h_1 the number of hyperplanes intersecting K in H_1 points and x the size of K . This yields the following equations

$$h_1 H_1 + \left(\frac{q^{2n+2} - 1}{q - 1} - h_1 \right) H_2 = x \frac{q^{2n+1} - 1}{q - 1}, \quad (5)$$

$$h_1 H_1 (H_1 - 1) + \left(\frac{q^{2n+2} - 1}{q - 1} - h_1 \right) H_2 (H_2 - 1) = x(x - 1) \frac{q^{2n} - 1}{q - 1}. \quad (6)$$

Solving h_1 in terms of x from the first equation and substituting this in the second equation yields a quadratic equation in x . The solutions are $x_1 = |Q^+(2n+1, q)|$ and x_2 , which is, as in the Hermitian case, an easily computed but tedious expression in q .

We show that the latter cannot occur. So suppose the set K contains x_2 points. Consider a space Π of codimension 2 intersecting the set K in C_i points. Denote by k_i the number of non-tangent hyperplanes containing Π . Then we obtain

$$k_i(H_1 - C_i) + (q + 1 - k_i)(H_2 - C_i) + C_i = x_2.$$

Solving this equation in k_i for $i = 1, 2, 3, 4$ yields

$$k_1 = 3 + \frac{q - 1}{q^n + 1}, \quad k_2 = q \frac{q^{n-1} + 1}{q^n + 1},$$

$$k_3 = \frac{2q^n + q + 1}{q^n + 1}, \quad k_4 = -\frac{q^{n+1} - 2q^n - 1}{q^n + 1}.$$

These are not natural numbers, the desired contradiction.

The second assertion follows by substituting $x = |Q^+(2n+1, q)|$ in Equation (5). ■

Lemma 2.9 *Through a space of codimension 2 of type C_1, C_2, C_3, C_4 there pass respectively $T_1 = 0, T_2 = 1, T_3 = 2$ and $T_4 = q + 1$ tangent hyperplanes.*

Proof Let Π be a codimension 2-space intersecting K in C_i points and let T_i denote the number of tangent hyperplanes containing Π . We obtain the following equation.

$$C_i + T_i(H_2 - C_i) + (q + 1 - T_i)(H_1 - C_i) = |K|.$$

Solving the equation in T_i for $i = 1, 2, 3$ yields the result. \blacksquare

Lemma 2.10 *Each tangent hyperplane contains A_i subspaces of codimension 2 intersecting K in C_i points, where*

$$A_1 = 0, \quad A_2 = q^{n-1}(q^n - 1), \quad A_3 = q^{2n},$$

$$A_4 = \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1}.$$

Proof Consider any tangent hyperplane Π . By the previous lemma $A_1 = 0$. Hence

$$\sum_{i=2}^4 A_i = \frac{q^{2n+1} - 1}{q - 1}.$$

We count in two ways the pairs (p, Δ) , $p \in \Delta \subset \Pi$, with p a point of K and Δ a subspace of codimension 2. We obtain

$$\sum_{i=2}^4 A_i C_i = H_2 \frac{q^{2n} - 1}{q - 1}.$$

Next we count in two ways the triples (p, r, Δ) , with $p, r \in \Delta \subset \Pi$, with $p \neq r$ points of K and Δ a subspace of codimension 2. We obtain

$$\sum_{i=2}^4 A_i C_i (C_i - 1) = H_2 (H_2 - 1) \frac{q^{2n-1} - 1}{q - 1}.$$

The obtained system of three linear equations in A_2, A_3 and A_4 can easily be solved, yielding the desired result. \blacksquare

Lemma 2.11 *Each point of K is contained in H_2 tangent hyperplanes, while each point not in K is contained in H_1 tangent hyperplanes.*

Proof We need to show that each point of K is contained in H_2 tangent hyperplanes. Set $K = \{p_1, \dots, p_{|Q^+(2n+1, q)|}\}$. Let a_i denote the number of tangent hyperplanes containing the point p_i . Counting pairs (p, τ) , $p \in K$, $p \in \tau$, τ a tangent hyperplane we obtain:

$$\sum_i a_i = |Q^+(2n+1, q)| H_2. \quad (7)$$

Next we count triples (p, τ_1, τ_2) , $p \in K$, $p \in \tau_i$, τ_i a tangent hyperplane, $i = 1, 2$. As before T_i denotes the number of tangent hyperplanes containing a fixed codimension 2-space Π of type C_i . We obtain the following equation.

$$\sum_i a_i(a_i - 1) = |Q^+(2n + 1, q)| \left(\sum_j A_j(T_j - 1)C_j \right). \quad (8)$$

From Equations (7) and (8) we can compute

$$|Q^+(2n + 1, q)| \sum_i a_i^2 - \left(\sum_i a_i \right)^2 = 0,$$

from which we deduce that a_i is a constant equal to H_2 .

The second assertion is proved in a similar way. \blacksquare

Denote by \mathcal{H} the set of tangent hyperplanes of K .

Let $\delta : \text{PG}(2n + 1, q) \rightarrow \text{PG}(2n + 1, q)^D$ be any fixed chosen duality of $\text{PG}(2n + 1, q)$.

Lemma 2.12 *The set $K' := \mathcal{H}^\delta$ is the point set of a non-singular hyperbolic quadric $Q^+(2n + 1, q)$ in $\text{PG}(2n + 1, q)^D$.*

Proof This is an immediate consequence of Lemma 2.9, Lemma 2.10 and Theorem 1.4. \blacksquare

Theorem 2.13 *The set K is the point set of a non-singular hyperbolic quadric $Q^+(2n + 1, q)$ in $\text{PG}(2n + 1, q)$.*

Proof Clearly K' is a point set satisfying the same conditions on intersections with hyperplanes and subspaces of codimension 2 as K in $\text{PG}(2n + 1, q)$. Since the tangent hyperplanes at K' are exactly those hyperplanes containing H_2 points of K' , we find that if we apply the duality δ^{-1} to $\text{PG}(2n + 1, q)^D$, the tangent hyperplanes of K' are mapped bijectively to the points of K (see Lemma 2.11). Hence we can now apply Lemma 2.12 with K' in $\text{PG}(2n + 1, q)^D$ replaced by K in $\text{PG}(2n + 1, q)$. We conclude that K is the point set of a $Q^+(2n + 1, q)$. \blacksquare

2.3 Elliptic quadrics

First of all let us recall what the intersections of a $Q^-(2n + 1, q)$ with hyperplanes and spaces of codimension 2 look like. A hyperplane of $\text{PG}(2n + 1, q)$ can intersect a $Q^-(2n + 1, q)$ either in a $Q(2n, q)$ or in a cone $pQ^-(2n - 1, q)$.

A space of codimension 2 can intersect a $Q^-(2n+1, q)$ either in a $Q^+(2n-1, q)$, a cone $pQ(2n-2, q)$, a $Q^-(2n-1, q)$ or a cone $LQ^-(2n-3, q)$.

So the intersection numbers with hyperplanes are

$$H_1 = \frac{q^{2n} - 1}{q - 1}, H_2 = 1 + q \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1}.$$

The intersection numbers with spaces of codimension 2 are

$$C_1 = \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1}, C_2 = 1 + q \frac{(q^{2n-2} - 1)}{q - 1},$$

$$C_3 = \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1}, C_4 = 1 + q + q^2 \frac{(q^{n-1} + 1)(q^{n-2} - 1)}{q - 1}.$$

From now on let K be a set of points in $\text{PG}(2n+1, q)$, $n \geq 2$ having the same intersection numbers with hyperplanes and codimension 2-spaces as a $Q^-(2n+1, q)$. We want to prove that K is the point set of a non-singular elliptic quadric. We will call subspaces intersecting K in a given number m of points, *subspaces of type m* . For obvious reasons a hyperplane intersecting K in H_2 points will also be called a *tangent hyperplane*.

Lemma 2.14 *The set K has size $|Q^-(2n+1, q)|$. Furthermore, there are $|Q^-(2n+1, q)|$ tangent hyperplanes.*

Proof This is proved in a similar way as Lemma 2.8. ■

Lemma 2.15 *Through a space of codimension 2 of type C_1, C_2, C_3, C_4 there pass respectively $T_1 = 0, T_2 = 1, T_3 = 2$ and $T_4 = q + 1$ tangent hyperplanes.*

Proof This is completely analogous to the proof of Lemma 2.9. ■

Lemma 2.16 *Each tangent hyperplane contains A_i subspaces of codimension 2 intersecting K in C_i points, where*

$$A_1 = 0, A_2 = q^{n-1}(q^n + 1), A_3 = q^{2n},$$

$$A_4 = \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1}.$$

Proof The proof is similar to the proof of Lemma 2.10. ■

Lemma 2.17 *Each point of K is contained in H_2 tangent hyperplanes, while each point not in K is contained in H_1 tangent hyperplanes.*

Proof This is proved as in Lemma 2.11. ■

Denote by \mathcal{H} the set of tangent hyperplanes of K . Let $\delta : \text{PG}(2n+1, q) \rightarrow \text{PG}(2n+1, q)^D$ be any fixed chosen duality of $\text{PG}(2n+1, q)$.

By Lemma 2.15 the set $K' := \mathcal{H}^\delta$ is a $(0, 1, 2, q+1)$ -set in $\text{PG}(2n+1, q)^D$. We want to show that K' is the point set of an elliptic quadric. Notice however that K' does not satisfy the conditions of Theorem 1.4. By Lemma 2.17 the intersection numbers of K' with respect to hyperplanes are H_1 and H_2 .

We define a point-line geometry S , with point set K' and line set those lines of $\text{PG}(2n+1, q)^D$ intersecting K' in $q+1$ points (the incidence is the natural one).

Theorem 2.18 *The geometry S is a Shult space such that no point of S is collinear with all other points of S .*

Proof Consider a point p of S and a line L of S , such that p and L are not incident. Consider the plane α generated by p and L .

If this plane contains another point r of S , then the fact that K' is a $(0, 1, 2, q+1)$ -set implies that α intersects S either in two intersecting lines or is fully contained in S (notice that $q > 2$ is necessary here). In both cases the “1-or-all axiom” holds.

Next suppose that α intersects S only in p and L . Assume that α would not be contained in a hyperplane of type H_1 . We count pairs (u, H) , with u a point of S not in α and H a hyperplane containing u and α . We obtain

$$\left(|Q^-(2n+1, q)| - q - 2\right) \frac{q^{2n-2} - 1}{q - 1} = \frac{q^{2n-1} - 1}{q - 1} (H_2 - q - 2),$$

a contradiction. Hence α is contained in at least one hyperplane Π of type H_1 . By Theorem 1.4 the $(0, 1, 2, q+1)$ -set $K' \cap \Pi$ is the point set of a non-singular parabolic quadric. We conclude that in S the point p is collinear with 1 or all points of L . Since each point of S is contained in at least one hyperplane of type H_1 no point of S is collinear with all other points of S . We still have to prove that through every point there passes a constant number of lines in order to conclude that S is a Shult space (using our definition from the introduction of Shult-space which is slightly different from the original one). Consider first two non-collinear points p and r . Then for every line containing p there is exactly one line through r . Hence through p and r there pass the same number of lines. Next consider two collinear points p and r . Considering all planes containing the line pr , it is clear that one can find a point s which is both non-collinear with p and r . By the above, there pass equally many lines through p and r . ■

Corollary 2.19 *The point set K' forms the point set of a non-singular elliptic quadric $Q^-(2n+1, q)$.*

Proof This is an immediate consequence of $|K'| = |Q^-(2n+1, q)|$, the previous lemma and Theorem 1.2. ■

Theorem 2.20 *The set K is the point set of a non-singular elliptic quadric $Q^-(2n+1, q)$ in $\text{PG}(2n+1, q)$.*

Proof Clearly K' is a point set satisfying the same conditions on intersections with hyperplanes and subspaces of codimension 2 as K in $\text{PG}(2n+1, q)$. Since the tangent hyperplanes at K' are exactly those hyperplanes containing H_2 points of K' , we find that if we apply the duality δ^{-1} to $\text{PG}(2n+1, q)^D$, the tangent hyperplanes of K' are mapped bijectively to the points of K (see Lemma 2.17). Hence we can now apply Corollary 2.19 with K' in $\text{PG}(2n+1, q)^D$ replaced by K in $\text{PG}(2n+1, q)$. We conclude that K is the point set of a $Q^-(2n+1, q)$. ■

2.4 Parabolic quadrics

First of all let us recall what the intersections of a $Q(2n, q)$ with hyperplanes and spaces of codimension 2 look like. A hyperplane can intersect a $Q(2n, q)$ either in a $Q^-(2n-1, q)$, a $Q^+(2n-1, q)$ or in a cone $pQ(2n-2, q)$.

A space of codimension 2 can intersect a parabolic quadric $Q(2n, q)$ either in a $Q(2n-2, q)$, a cone $pQ^+(2n-3, q)$, a cone $pQ^-(2n-3, q)$ or a cone $LQ(2n-4, q)$ (notice that this cone contains the same number of points as a $Q(2n-2, q)$). So the intersection numbers with hyperplanes are

$$H_1 = \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1}, H_2 = \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1},$$

$$H_3 = 1 + q \frac{q^{2n-2} - 1}{q - 1}.$$

The intersection numbers with spaces of codimension 2 are

$$C_1 = \frac{q^{2n-2} - 1}{q - 1}, C_2 = 1 + q \frac{(q^{n-1} - 1)(q^{n-2} + 1)}{q - 1},$$

$$C_3 = 1 + q \frac{(q^{n-1} + 1)(q^{n-2} - 1)}{q - 1}.$$

From now on let K be a set of points in $\text{PG}(2n, q)$ having the same intersection numbers with hyperplanes and codimension 2-spaces as a $Q(2n, q)$. We want to prove that K is the point set of a $Q(2n, q)$.

We will call subspaces intersecting K in a given number m of points, *subspaces of type m* . For obvious reasons a hyperplane intersecting K in H_2 points will also be called a *tangent hyperplane*.

Lemma 2.21 *The set K contains $|Q(2n, q)|$ points.*

Proof Let h_i , respectively c_i , denote the number of hyperplanes of type H_i , respectively the number of codimension 2 spaces of type C_i . By counting pairs and triples as in Lemma 2.8, but now with respect to hyperplanes as well as with respect to codimension 2 spaces, we obtain

$$\sum_i h_i = \frac{q^{2n+1} - 1}{q - 1}, \quad (9)$$

$$\sum_i h_i H_i = \frac{q^{2n} - 1}{q - 1} |K|, \quad (10)$$

$$\sum_i h_i H_i (H_i - 1) = \frac{q^{2n-1} - 1}{q - 1} |K| (|K| - 1), \quad (11)$$

$$\sum_i c_i = \frac{(q^{2n+1} - 1)(q^{2n} - 1)}{(q^2 - 1)(q - 1)}, \quad (12)$$

$$\sum_i c_i C_i = |K| \frac{(q^{2n} - 1)(q^{2n-1} - 1)}{(q^2 - 1)(q - 1)}, \quad (13)$$

$$\sum_i c_i C_i (C_i - 1) = |K| (|K| - 1) \frac{(q^{2n-1} - 1)(q^{2n-2} - 1)}{(q^2 - 1)(q - 1)}. \quad (14)$$

Now consider a hyperplane Π of type H_j and denote by m_i^j the number of codimension 2 spaces of type C_i it contains. By counting pairs (p, Δ) and triples (p, r, Δ) , with $p \neq r$ points of $K \cap \Pi$, $\Delta \subset \Pi$ a codimension 2 space, and $p, r \in \Delta$, we obtain

$$\sum_i m_i^j = \frac{q^{2n} - 1}{q - 1}$$

$$\sum_i m_i^j C_i = H_j \frac{q^{2n-1} - 1}{q - 1}$$

$$\sum_i m_i^j C_i (C_i - 1) = H_j (H_j - 1) \frac{q^{2n-2} - 1}{q - 1}$$

From these equations all values m_i^j can easily be determined. As $m_2^2 = 0$ we see that the number of hyperplanes of type H_1 through a

codimension 2 space of type C_2 is a constant $f(K)$ depending only on the size of K , with $f(K)$ linear in $|K|$.

By counting pairs (H, Δ) , with H a hyperplane of type H_1 , $\Delta \subset H$ a codimension 2 space of type C_2 , we obtain

$$h_1 m_2^1 = c_2 f(K). \quad (15)$$

So $h_1 = h_1(c_2, K)$, with $h_1(c_2, K)$ linear in $|K|$. From Equations (12), (13) and (14) we can obtain an expression for c_2 depending only on $|K|$, say $c_2 = c_2(K)$, with $c_2(K)$ quadratic in $|K|$. From Equations (9), (10) and (11) we can now obtain a quadratic equation in $|K|$ of the form $s|K|^2 + t|K| + h_1 = 0$. By substitution we obtain

$$s|K|^2 + t|K| + h_1(c_2(K), K) = 0, \quad (16)$$

which turns out to be a cubic equation in $|K|$. One simply checks that $|K| = \frac{q^{2n}-1}{q-1}$ is a solution of Equation (16).

We want to exclude the other roots of Equation (16) as possible sizes for the set K .

Though Maple is not able to directly calculate the other roots for general n and q it is not too hard to determine the product and sum of the roots of Equation (16) with it.

One obtains that the three roots have product

$$\frac{q^{4n-2} + q^{2n+1} - 3q^{2n} + q^{2n-1} - q^{2n-2} + 1}{(q-1)^3 (q^{2n-1} - 1)},$$

and sum

$$3 \frac{(q^n + 1)(q^n - 1)}{q - 1}.$$

From the above expressions one can deduce that the roots different from $\frac{q^{2n}-1}{q-1}$ are not real numbers, a contradiction.

Lemma 2.22 *There are exactly $\frac{q^{2n}-1}{q-1}$ tangent hyperplanes. Furthermore, every codimension 2 space of type C_2 or C_3 is contained in exactly one tangent hyperplane.*

Proof The first assertion follows from Equations (9), (10) and (11) once we know $|K| = \frac{q^{2n}-1}{q-1}$. To prove the second assertion, we notice that $m_3^1 = m_2^2 = 0$. Hence, if T_i denotes the number of tangent hyperplanes containing a given codimension 2 space of type C_i , $i = 2, 3$, then

$$T_i(H_3 - C_i) + (q + 1 - T_i)(H_{i-1} - C_i) + C_i = |K|.$$

We obtain $T_2 = T_3 = 1$. ■

Lemma 2.23 For every codimension 2-space of type C_1 the number of hyperplanes of type H_1 containing it is equal to the number of hyperplanes of type H_2 in which it is contained.

Proof One notices that $|K| = (q+1)(H_3 - C_1) + C_1$ and that $(H_1 + H_2)/2 = H_3$. The lemma follows. ■

Lemma 2.24 Let γ be a codimension 3 space contained in a hyperplane H of type H_1 . Suppose that γ is contained in N_H codimension 2 spaces α of type C_2 , such that $\gamma \subset \alpha \subset H$. Then $|\gamma \cap K| = q^{n-2}N_H + \frac{(q^{n-1}+1)(q^{n-2}-1)}{q-1}$. Furthermore if γ is also contained in a hyperplane E of type H_2 , then $N_H \leq 2$.

Proof Let X denote the number of points of K contained in γ . As $m_3^1 = 0$ we have

$$(q+1 - N_H)(C_1 - X) + N_H(C_2 - X) + X = H_1.$$

It follows that

$$X = N_H q^{n-2} + \frac{(q^{n-1} + 1)(q^{n-2} - 1)}{q - 1}.$$

Next let E be a hyperplane of type H_2 containing γ and let N_E be the number of codimension 2 spaces β of type C_3 such that $\gamma \subset \beta \subset E$. Since $m_2^2 = 0$ we obtain that

$$(q+1 - N_E)(C_1 - X) + N_E(C_3 - X) + X = H_2.$$

Substitution of the higher obtained expression for X in terms of N_H yields

$$N_E = 2 - N_H.$$

As $N_E \geq 0$ the lemma follows. ■

Lemma 2.25 A codimension 3 space that is contained in a hyperplane of type H_1 contains $Nq^{n-2} + \frac{(q^{n-1}+1)(q^{n-2}-1)}{q-1}$ points of K , with $N \in \{0, 1, 2, q+1\}$.

Proof Let γ be any codimension 3 space contained in a hyperplane of type H_1 , and set $X = |\gamma \cap K|$. If γ is contained in hyperplanes of type H_1 as well as of type H_2 there is nothing to prove because of the previous lemma.

If γ is contained in no hyperplane of type H_2 , then γ cannot be contained in a codimension 2 space of type C_3 (since through each codimension 2 space of type C_3 there passes a hyperplane of type H_2 ,

which follows from $m_3^1 = 0$). Furthermore, since the number of H_2 hyperplanes containing a given codimension 2 space of type C_1 equals the number of H_1 hyperplanes containing it, γ cannot be contained in a codimension 2 space α of type C_1 such that $\gamma \subset \alpha \subset H$. Hence in H all codimension 2 spaces containing γ must be of type C_2 . By the previous lemma we obtain that $X = (q+1)q^{n-2} + \frac{(q^{n-1}+1)(q^{n-2}-1)}{q-1}$. ■

Corollary 2.26 *Every hyperplane of type H_1 intersects K in the point set of a non-singular hyperbolic quadric.*

Proof By the previous lemma the conditions of Theorem 2.13 are satisfied in each hyperplane of type H_1 , whenever $n > 2$. If $n = 2$, the corollary follows by the remark after Theorem 1.6. ■

Lemma 2.27 *Every point of K is contained in at least one hyperplane of type H_1 .*

Proof Let p be any point of K . Assume by way of contradiction that p is contained only in hyperplanes of type H_2 and H_3 . Then, by counting pairs (r, H) , $r \in K$, $r \neq p$, $p, r \in H$, H a hyperplane, we obtain

$$l_2(H_2 - 1) + \left(\frac{q^{2n} - 1}{q - 1} - l_2 \right) (H_3 - 1) = \left(\frac{q^{2n} - 1}{q - 1} - 1 \right) \frac{q^{2n-1} - 1}{q - 1},$$

with l_2 the number of hyperplanes of type H_2 containing p . It follows that $l_2 < 0$, an absurdity. ■

Theorem 2.28 *The set K is the point set of a non-singular parabolic quadric.*

Proof Let L be any line of $\text{PG}(2n, q)$, and suppose that $|L \cap K| = x$, with $x \geq 3$. Suppose there would be no hyperplane of type H_1 containing L . Then, if n_2 would be the number of hyperplanes of type H_2 containing L , we obtain by counting pairs (r, H) , $r \in K$, $r \notin L$, H a hyperplane containing r and L

$$n_2(H_2 - x) + \left(\frac{q^{2n-1} - 1}{q - 1} - n_2 \right) (H_3 - x) = \left(\frac{q^{2n} - 1}{q - 1} - x \right) \frac{q^{2n-2} - 1}{q - 1}.$$

This implies that $n_2 < 0$, a contradiction. Hence L is contained in a hyperplane of type H_1 . Corollary 2.26 implies that $x = q + 1$. Consequently K is a $(0, 1, 2, q + 1)$ -set in $\text{PG}(2n, q)$. By combining Lemma 2.27 and Corollary 2.26 we also see that each point of K is contained in a line M such that $|M \cap K| = 2$. Hence K is non-singular. It follows that K satisfies the conditions of Theorem 1.4. As $|K| = \frac{q^{2n}-1}{q-1}$ the theorem follows. ■

The Main Theorem 1.6 is now an immediate consequence of Theorems 2.7, 2.13, 2.20 and 2.28.

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