

# Characterizations of hermitian varieties by intersection numbers

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## Abstract

In this paper, we give characterizations of the classical generalized quadrangles  $H(3, q^2)$  and  $H(4, q^2)$ , embedded in  $PG(3, q^2)$  and  $PG(4, q^2)$ , respectively. The intersection numbers with lines and planes characterize  $H(3, q^2)$ , and  $H(4, q^2)$  is characterized by its intersection numbers with planes and solids.

This result is then extended to characterize all hermitian varieties in dimension at least 4 by their intersection numbers with planes and solids.

## 1 Introduction

In the early nineties Tallini and Ferri gave a characterization of the parabolic quadric  $Q(4, q)$  by its intersection numbers with planes and solids [9]. Here we will characterize the hermitian varieties  $H(3, q^2)$  and  $H(4, q^2)$ , embedded in  $PG(3, q^2)$  and  $PG(4, q^2)$ , respectively. The intersection numbers with lines and planes will characterize  $H(3, q^2)$ , while the intersection numbers with planes and solids will characterize  $H(4, q^2)$ . Finally, all point sets of  $PG(n, q^2)$ ,  $n \geq 4$ , having only intersection numbers with planes and solids which are also intersection numbers of hermitian varieties in  $n$  dimensions with planes and solids are determined.

## 2 Notation and background on polar spaces and generalized quadrangles

Polar spaces were first described axiomatically by Veldkamp [14]. Later on, Tits simplified Veldkamp's list of axioms and further completed the theory [13]. We recall Tits' definition of polar spaces.

A *polar space of rank  $n$* ,  $n \geq 2$ , is a point set  $P$  together with a family of subsets of  $P$  called *subspaces*, satisfying the following axioms.

- (i) A subspace, together with the subspaces it contains, is a  $d$ -dimensional projective space with  $-1 \leq d \leq n-1$ ,  $d$  is called the dimension of the subspace.
- (ii) The intersection of two subspaces is a subspace.
- (iii) Given a subspace  $V$  of dimension  $n-1$  and a point  $p \in P \setminus V$ , there is a unique subspace  $W$  of dimension  $n-1$  such that  $p \in W$  and  $V \cap W$  has dimension  $n-2$ ;  $W$  contains all points of  $V$  that are joined to  $p$  by a subspace of dimension 1, also called a line.
- (iv) There exist two disjoint subspaces of dimension  $n-1$ .

The finite classical polar spaces are the following structures.

- (i) The non-singular quadrics in odd dimension,  $Q^+(2n+1, q)$ ,  $n \geq 1$ , and  $Q^-(2n+1, q)$ ,  $n \geq 2$ , together with the subspaces they contain, give a polar space of rank  $n+1$  and  $n$ , respectively. The non-singular parabolic quadrics  $Q(2n, q)$ ,  $n \geq 2$ , in even dimension, together with the subspaces they contain, give a polar space of rank  $n$ .
- (ii) The non-singular hermitian varieties in  $PG(2n, q^2)$ ,  $n \geq 2$ , together with the subspaces they contain, give a polar space of rank  $n$ ; the non-singular hermitian varieties in  $PG(2n+1, q^2)$ ,  $n \geq 1$ , together with the subspaces they contain, give a polar space of rank  $n+1$ .
- (iii) The points of  $PG(2n+1, q)$ ,  $n \geq 1$ , together with the totally isotropic subspaces of a non-singular symplectic polarity of  $PG(2n+1, q)$ , give a polar space of rank  $n+1$ .

By theorems of Veldkamp and Tits, all polar spaces with finite rank at least 3 are classified. In the finite case, i.e. the polar space has a finite number of points, the following theorem, which can be found in [13], holds.

**Theorem 2.1** *A finite polar space of rank at least 3 is classical.*

Buekenhout and Shult described polar spaces as point-line geometries, and this description will be used in this paper.

**Definition** A *Shult space* is a point-line geometry  $S = (P, B, I)$ , with  $B$  a non-empty set of subsets of  $P$  of cardinality at least 2, such that the incidence relation  $I$ , which is containment here, satisfies the following axiom. For each line  $L \in B$  and for each point  $p \in P \setminus L$ , the point  $p$  is collinear with either one or all points of the line  $L$ .

A Shult space is *non-degenerate* if no point is collinear with all other points.

A *subspace* of a Shult space  $S = (P, B, I)$  is a subset  $W$  of  $P$  such that any two points of  $W$  are on a common line and any line containing distinct points of  $W$  is completely contained in  $W$ . A Shult space is *linear* if two distinct lines have at most one common point. Buekenhout and Shult proved the following fundamental theorem [8].

**Theorem 2.2** (i) *Every non-degenerate Shult space is linear.*

(ii) *If  $S$  is a non-degenerate Shult space of finite rank at least 3, and if all lines contain at least three points, then the Shult space together with all its subspaces is a polar space.*

A *finite generalized quadrangle* ( $GQ$ ) of order  $(s, t)$  is an incidence structure  $S = (P, B, I)$  in which  $P$  and  $B$  are disjoint non-empty sets of objects called *points* and *lines* respectively, and for which  $I$  is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with  $t + 1$  lines ( $t \geq 1$ ) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with  $s + 1$  points ( $s \geq 1$ ) and two distinct lines are incident with at most one point.

(GQ3) If  $p$  is a point and  $L$  is a line not incident with  $p$ , then there is a unique point-line pair  $(q, M)$  such that  $pIMqIL$ .

A generalized quadrangle ( $GQ$ ) of order  $(s, t)$  contains  $(s + 1)(st + 1)$  points, see [12]. If  $s = t$ , then  $S$  is also said to be of order  $s$ .

If  $S$  has a finite number of points and if  $s > 1$ , it is easy to show one can replace axiom (GQ1) by the following axioms.

(GQ1') No point is collinear with all others.

(GQ1'') There is a point on at least two lines.

It is this alternative definition which we will use in our proofs.

## 2.1 The classical generalized quadrangles

Consider a non-singular quadric of Witt index 2, that is of projective index 1, in  $PG(3, q)$ ,  $PG(4, q)$  and  $PG(5, q)$ . The points and lines of these quadrics form generalized quadrangles which are denoted by  $Q^+(3, q)$ ,  $Q(4, q)$  and  $Q^-(5, q)$ , and of order  $(q, 1)$ ,  $(q, q)$  and  $(q, q^2)$ . Next, let  $H$  be a non-singular hermitian variety in  $PG(3, q^2)$  or  $PG(4, q^2)$ . The points and lines of  $H$  form a generalized quadrangle  $H(3, q^2)$  or  $H(4, q^2)$ , which has order  $(q^2, q)$  or  $(q^2, q^3)$ . The points of  $PG(3, q)$  together with the totally isotropic lines with respect to a symplectic polarity form a  $GQ$ , denoted by  $W(q)$ , of order  $q$ . The generalized quadrangles defined here are the so-called classical generalized quadrangles.

**Definition** Let  $V$  be a vector space over some skew field, not necessarily finite-dimensional. A generalized quadrangle  $S = (P, B, I)$  is *fully embedded* in the projective space  $PG(V)$  if there is a map  $\pi$  from  $P$  to the set of points and from  $B$  to the set of lines of  $PG(V)$  such that:

- (i)  $\pi$  is injective on points,
- (ii) if  $x \in P$  and  $L \in B$  with  $xIL$ , then  $x^\pi \in L^\pi$ ,
- (iii) the set of points  $x^\pi$ , where  $x \in P$ , generates  $PG(V)$ ,
- (iv) every point in  $PG(V)$  on the image of a line of the quadrangle is also the image of a point of the quadrangle.

The following beautiful theorem is due to Buekenhout and Lefèvre [7].

**Theorem 2.3** *Every finite generalized quadrangle fully embedded in projective space is classical.*

A lot of information on finite generalized quadrangles can be found in the reference work [12].

### 3 Notation and background on blocking sets

A *blocking set*  $B$  in  $\Pi = PG(2, q)$  is a set of points of  $\Pi$  which meets every line. A line is an example of a blocking set, but a blocking set containing a line is called *trivial*.

A blocking set is called *minimal* if for every  $p \in B$ , the point set  $B - \{p\}$  is not a blocking set. It is easy to prove the following useful lemma.

**Lemma 3.1** *A blocking set  $B$  is minimal if and only if for every point  $p$  of  $B$ , there is some line  $L$  such that  $B \cap L = \{p\}$ .*

A blocking set containing  $k$  points is called a *blocking  $k$ -set*. The following theorem gives an upper and a lower bound on the size of a non-trivial minimal blocking set.

**Theorem 3.2** *Let  $B$  be a non-trivial minimal blocking set in  $PG(2, q)$ . Then*

- (i) [4]  $|B| \geq q + \sqrt{q} + 1$  with equality if and only if  $q$  is a square and  $B$  is a Baer subplane.
- (ii) [6]  $|B| \leq q\sqrt{q} + 1$  with equality if and only if  $q$  is a square and  $B$  is a unital.

Next, let us consider multiple blocking sets.

**Definition** An  $s$ -fold blocking set in  $PG(2, q)$  is a set of points of  $PG(2, q)$  that intersects every line in at least  $s$  points. It is called *minimal* if no proper subset is an  $s$ -fold blocking set.

A 1-fold blocking set is a blocking set. The following theorem shows that if  $s > 1$ , then in order to find  $s$ -fold blocking sets of small cardinality one must look for sets not containing a line.

**Theorem 3.3** Let  $B$  be an  $s$ -fold blocking set of  $PG(2, q)$ ,  $s > 1$ .

- (i) [5] If  $B$  contains a line, then  $|B| \geq sq + q - s + 2$ .
- (ii) [1] If  $B$  does not contain a line, then  $|B| \geq sq + \sqrt{sq} + 1$ .

For small  $s$  this theorem can be improved and if  $s$  is small and  $q$  is a square the smallest minimal  $s$ -fold blocking sets are classified [2]. Finally we introduce blocking sets in higher dimensional spaces.

**Definition** A blocking set with respect to  $t$ -spaces in  $PG(n, q)$  is a set  $B$  of points such that every  $t$ -dimensional subspace of  $PG(n, q)$  meets  $B$  in at least one point.

The following result by Bose and Burton gives a nice characterization of the smallest ones [3].

**Theorem 3.4** If  $B$  is a blocking set with respect to  $t$ -spaces in  $PG(n, q)$  then  $|B| \geq |PG(n - t, q)|$  and equality holds if and only if  $B$  is an  $(n - t)$ -dimensional subspace.

## 4 Characterizations of $H(3, q^2)$ and $H(4, q^2)$

### 4.1 The classical generalized quadrangle $H(3, q^2)$

It is known that a non-singular hermitian variety  $H(3, q^2)$  in  $PG(3, q^2)$  intersects lines either in a point, a Baer subline or all  $q^2 + 1$  points of the line. So the intersection numbers with lines are 1,  $q + 1$ , and  $q^2 + 1$ . It intersects planes either in a non-singular hermitian variety  $H(2, q^2)$  or in a cone with vertex a point and as base a Baer subline. So the intersection numbers with planes are  $q^3 + 1$  and  $q^3 + q^2 + 1$ .

The following terminology is introduced for the considered set  $K$ ; it is also used in the next section. Lines intersecting  $K$  in one point are *tangent* lines, lines intersecting  $K$  in  $q + 1$  points *Baer* lines, and lines intersecting  $K$  in  $q^2 + 1$  points *full* lines.

Planes that intersect our set  $K$  in  $q^3 + 1$  points are called *non-singular* planes, and planes that intersect  $K$  in  $q^3 + q^2 + 1$  points are called *singular* planes. Denote the intersection of a plane  $\alpha$  and the set  $K$  by  $K_\alpha$ .

**Theorem 4.1** *If the intersection numbers with lines and planes of the point set  $K$  of  $PG(3, q^2)$  are intersection numbers with lines and planes of  $H(3, q^2)$ , then  $K$  is a hermitian variety  $H(3, q^2)$ .*

**Proof** Firstly, every plane  $\alpha$  contains a tangent line. Suppose the contrary; then every line of  $\alpha$  intersects  $K$  in at least  $q + 1$  points. This means that  $K_\alpha$  is a  $(q + 1)$ -fold blocking set in  $\alpha$ . By Theorem 3.3(ii)  $(q + 1)$ -fold blocking sets have size at least  $(q + 1)q^2 + \sqrt{(q + 1)q^2} + 1$ , which yields a contradiction with the assumptions on the intersection numbers with planes.

Next, there exists a full line. Suppose this is not the case, then we distinguish two cases.

Either no singular plane exists in which case all planes are non-singular. In this case all lines must have the same intersection number with the set  $K$ . Suppose the contrary, so assume that there exists at least one tangent line, and at least one Baer line, and consider all planes in  $PG(3, q^2)$  through a line  $L$ , respectively  $M$ , where we take  $L$  to be a tangent line and  $M$  to be a Baer line. This yields two different numbers for the size of  $K$ , a contradiction. Since all planes contain at least one tangent line, this means that all lines are tangent ones. Then consider a plane  $\alpha$  and a point  $p$  of  $K$  inside  $\alpha$ . Look at all lines through  $p$  inside  $\alpha$ . Then  $\alpha$  would contain only the point  $p$ , a contradiction.

The other case is that there exists a singular plane  $\alpha$ . Take a point  $p$  of  $K_\alpha$  lying on a tangent line  $L$  and consider all lines through  $p$  inside  $\alpha$ . After counting we find at most  $1 + (q^2)q = 1 + q^3$  points, again a contradiction. So full lines certainly exist.

Also singular planes exist. To show this, consider a full line  $L$  and a point  $p$  of  $K$  which is not incident with  $L$ . Consider all lines through  $p$  in the plane  $\alpha$  generated by  $p$  and  $L$ . All these lines contain at least 2 points hence they are Baer lines or full lines. So at least  $1 + q(q^2 + 1) = q^3 + q + 1$  points of the set  $K$  belong to  $\alpha$ . Hence,  $\alpha$  necessarily is a singular plane, and elementary counting yields that exactly one of the lines through  $p$  inside  $\alpha$  is a full line, say  $M$ .

This line  $M$  intersects  $L$  in a point  $r$ . Consider a point  $p'$  in  $K_\alpha$  not lying on  $L$  or  $M$ . Through this point  $p'$  there goes a full line  $N$  in  $\alpha$  by the previous. The line  $N$  necessarily contains  $r$ , otherwise it intersects  $L$  and  $M$  in two different points  $s$  and  $t$  respectively, but then through  $s \notin M$  there are two full lines in  $\alpha$  that intersect  $M$ , namely  $L$  and  $N$ , a contradiction. Hence, all lines through  $r$  in  $\alpha$  are either tangent lines or full lines, so counting yields that inside  $\alpha$  there pass  $q + 1$  full lines through  $r$ .

One can also calculate the size of  $K$ . Consider all planes through a full line; since they are all singular by the previous arguments, this yields

$$|K| = (q^2 + 1)q^3 + q^2 + 1 = (q^3 + 1)(q^2 + 1).$$

Define  $S$  to be the incidence structure with as points the points of the set  $K$ , as lines the full lines, and where a point  $p$  and a line  $L$  are said to be incident if  $L$  passes through  $p$ . We have already proved that axioms (GQ2) and (GQ3) for generalized quadrangles hold for  $S$ .

It is impossible that one point  $p$  of  $S$  is collinear with all other points of  $S$ . If this were the case, then consider a plane  $\pi$  not through  $p$ . Since this plane intersects  $K$  in either  $q^3 + 1$  or  $q^3 + q^2 + 1$  points,  $K$  would contain either  $1 + q^2(q^3 + 1)$  or  $1 + q^2(q^3 + q^2 + 1)$  points, which yields a contradiction with the size of  $K$ .

Since it was proved that there exists a point  $p$  in  $S$  such that there go at least 3 lines of  $S$  through  $p$ , and since no point of  $S$  is collinear with all other points of  $S$ , we have shown that the incidence structure  $S$  is a generalized quadrangle. Hence by the theorem of Buekenhout and Lefèvre it is a classical one, and by looking at the different classical generalized quadrangles, it has to be  $H(3, q^2)$ .  $\square$

## 4.2 The classical generalized quadrangle $H(4, q^2)$

Consider a set of points  $K$  in  $PG(4, q^2)$ , such that all intersection numbers with planes and solids are also intersection numbers with planes and solids of the non-singular hermitian variety  $H(4, q^2)$ . We show that  $K$  has to be  $H(4, q^2)$ . Since  $H(4, q^2)$  intersects a plane either in a line, a non-singular hermitian variety  $H(2, q^2)$ , or a cone with vertex a point  $p$  and base a Baer subline  $H(1, q^2)$ , the intersection numbers with planes are

$$q^2 + 1, q^3 + 1, q^3 + q^2 + 1.$$

Call planes with intersection number either  $q^2 + 1$ ,  $q^3 + 1$  or  $q^3 + q^2 + 1$ , *small*, *medium* or *large*.

The intersection of a non-singular hermitian variety  $H(4, q^2)$  with a solid is either a non-singular hermitian variety  $H(3, q^2)$  or a cone with vertex a point  $p$  and base a non-singular hermitian variety  $H(2, q^2)$ , so the intersection numbers with solids are

$$(q^2 + 1)(q^3 + 1), q^2(q^3 + 1) + 1.$$

Call solids with intersection number  $q^2(q^3 + 1) + 1$  *singular*, the other ones *non-singular*. For a given plane  $\alpha$ , the set of points belonging to  $K \cap \alpha$  will be denoted by  $K_\alpha$ . A line  $L$  intersecting  $K$  in  $q^2 + 1$  points is called a *full* line.

**Theorem 4.2** *If each intersection number with planes and solids of a point set  $K$  in  $PG(4, q^2)$  is also an intersection number with planes and solids of  $H(4, q^2)$ , then  $K$  is a non-singular hermitian variety  $H(4, q^2)$ .*

To prove Theorem 4.2 we first need a series of lemmas.

**Lemma 4.3** *The set  $K$  contains  $|H(4, q^2)|$  points.*

**Proof** Call  $H_1 = q^2(q^3 + 1) + 1$  the number of points of  $K$  contained in a singular solid,  $H_2 = (q^2 + 1)(q^3 + 1)$  the number of points of  $K$  contained in a non-singular solid, and  $x$  the total number of points contained in the set  $K$ . Call  $a$  the number of singular solids. Then counting the pairs  $(p, \alpha)$  where  $p$  is a point and  $\alpha$  a solid such

that  $p \in K \cap \alpha$ , respectively the triples  $(p, r, \alpha)$  with  $p, r \in K \cap \alpha$ ,  $p \neq r$ , in two ways yields the following equations

$$aH_1 + \left(\frac{q^{10} - 1}{q^2 - 1} - a\right)H_2 = x\frac{q^8 - 1}{q^2 - 1},$$

$$aH_1(H_1 - 1) + \left(\frac{q^{10} - 1}{q^2 - 1} - a\right)H_2(H_2 - 1) = x(x - 1)\frac{q^6 - 1}{q^2 - 1}.$$

From the first equation we obtain  $a$  in function of  $x$ . Substituting this in the second equation yields a quadratic equation in  $x$ . This equation has the following two solutions

$$x = q^7 + q^5 + q^2 + 1, \quad \frac{q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1}{q^2 + q + 1}.$$

The first solution is the desired one. The second one is not a natural number, this proves the lemma.  $\square$

Consider a small plane  $\alpha$  and look at all solids through  $\alpha$  inside  $PG(4, q^2)$ . Call the number of singular and non-singular ones  $a$  and  $b$  respectively, so  $a + b = q^2 + 1$ . An elementary counting yields the following equation

$$aq^5 + b(q^5 + q^3) + q^2 + 1 = (q^5 + 1)(q^2 + 1).$$

Solving these equations yields  $a = q^2 + 1$  and  $b = 0$ . So all solids containing a small plane are singular. Similar calculations learn that the other planes are contained in both kinds of solids.

**Lemma 4.4** *There exist solids of both kinds.*

**Proof** Suppose the contrary. Then the calculations above show that all planes are small and all solids are singular. Take a line  $L$  containing  $c$  points of the set  $K$ . Consider all planes through  $L$  inside a solid. One gets the following equation,

$$(q^2 + 1)(q^2 + 1 - c) + c = q^5 + q^2 + 1.$$

This yields a contradiction for all  $c$ . Hence, there exist solids of both kinds.  $\square$

**Lemma 4.5** *For a large plane  $\alpha$ , the set  $K_\alpha$  is a blocking set in  $\alpha$ .*

**Proof** Suppose that  $\alpha$  contains a line  $L$  which is not blocked by  $K_\alpha$ . Consider a large solid  $\Pi$  through  $\alpha$ . Since through small planes, there only pass singular solids, all planes through  $L$  inside  $\Pi$  are medium or large ones, but then  $\Pi$  intersects the set  $K$  in more than  $(q^2 + 1)(q^3 + 1)$  points, which yields a contradiction.  $\square$

Next we prove that also for small and medium planes  $\alpha$  the sets  $K_\alpha$  are blocking sets in  $\alpha$ . The case  $q = 2$  will be proved separately. First, the general case is proved.



**Lemma 4.6** *Consider a medium plane  $\alpha$ . If  $q > 2$ , then  $K_\alpha$  is a blocking set in  $\alpha$ .*

**Proof** Suppose that  $K_\alpha$  is not a blocking set in  $\alpha$ , meaning that  $\alpha$  contains a line  $L$  which is not blocked by  $K_\alpha$ . Consider a singular solid  $\Pi$  through  $\alpha$ , and look at all planes through  $L$  inside  $\Pi$ . By the previous lemma, all these planes are small or medium. Call  $a$  the number of small ones, and  $b$  the number of medium ones. We obtain the following equation

$$a(q^2 + 1) + b(q^3 + 1) = q^2(q^3 + 1) + 1.$$

Substituting  $a = q^2 + 1 - b$  in the above expression yields

$$b = q^2 - \frac{1}{q-1}.$$

Since  $b$  has to be a natural number, this is a contradiction if  $q > 2$ .  $\square$

**Lemma 4.7** *Consider a small plane  $\alpha$ . If  $q > 2$ , then  $K_\alpha$  is a blocking set in  $\alpha$  and hence the points belonging to  $K_\alpha$  form a line of  $\alpha$ .*

**Proof** If  $\alpha$  contains a line  $L$  which is not blocked by the set  $K_\alpha$ , then by the previous lemmas all planes through  $L$  inside a solid  $\Pi$  on  $\alpha$  must be small ones. But then  $\Pi$  intersects  $K$  in  $(q^2 + 1)^2$  points, which yields a contradiction.

So  $K_\alpha$  is a blocking set in  $\alpha$ , and blocking sets of size  $q^2 + 1$  in a plane of order  $q^2$  are lines by the Bose-Burton theorem.  $\square$

Next, the case  $q = 2$ .

**Lemma 4.8** *Let  $q = 2$  and consider a plane  $\alpha$  that is either small or medium. Then the set  $K_\alpha$  is a blocking set for  $\alpha$ . In particular, a small plane  $\alpha$  intersects  $K$  in a line.*

**Proof** Suppose that a small plane  $\alpha$  contains a line  $L$  which is not blocked by  $K_\alpha$ . Consider a singular solid  $\Pi$  containing  $\alpha$ , and call the number of small and medium planes through  $L$  inside  $\Pi$ ,  $\phi$  and  $\psi$  respectively. By Lemma 3.4 we know that  $\phi + \psi = q^2 + 1$ . Counting the number of intersection points of  $\Pi$  and  $K$  yields the following equation

$$\phi(q^2 + 1) + \psi(q^3 + 1) = q^2(q^3 + 1) + 1.$$

Substituting  $q = 2$  into these equations, one obtains  $\phi = 2$  and  $\psi = 3$ .

Next it is shown that the total number of small planes inside  $\Pi$  is 2. Call the number of small, medium and large planes inside  $\Pi$ ,  $a$ ,  $b$  and  $c$  respectively. Then by counting the total number of planes in a solid, the incident pairs  $(p, \alpha)$ , where  $p$  is a point of  $K$  and  $\alpha$  a plane of  $\Pi$ , and the triples  $(p, r, \alpha)$  where  $p$  and  $r$  are distinct points of

$K$  and  $\alpha$  is a plane in  $\Pi$  containing  $p$  and  $r$ , in two ways, the following equations are obtained, where  $k = q^2(q^3 + 1) + 1$ ,

$$a + b + c = (q^2 + 1)(q^4 + 1),$$

$$a(q^2 + 1) + b(q^3 + 1) + c(q^3 + q^2 + 1) = k(q^4 + q^2 + 1),$$

$$a(q^2 + 1)q^2 + b(q^3 + 1)q^3 + c(q^3 + q^2 + 1)(q^3 + q^2) = k(k - 1)(q^2 + 1).$$

Solving these equations yields  $a = q(q - 1)$ , so if  $q = 2$  then  $a = 2$ . This implies that all other lines inside  $\alpha$  are blocked, otherwise at least 3 small planes are contained in  $\Pi$ , since each non-blocked line has two small planes through it in  $\Pi$  and only  $\alpha$  can be counted twice.

This implies that by adding an arbitrary point  $r$  on  $L$  to  $K_\alpha$  a blocking set of size  $q^2 + 2$  arises and so each of these sets has to contain a line  $M_r$  by Theorem 3.2. Since the point  $r$  on  $L$  was arbitrary this means all these sets contain the same line  $M$  already contained in  $K_\alpha$ , but then  $L$  is blocked by  $K_\alpha$ , a contradiction.

Consider a line  $L$  in a medium plane  $\alpha$ . Consider a small solid  $\Pi$  containing  $\alpha$ . The calculation in the beginning of the proof shows that  $L$  is contained in a small plane. Hence,  $L$  is blocked.  $\square$

**Remark:** It was already shown that singular solids exist, and by the previous proof all singular solids contain small planes. Since small planes intersect  $K$  in a full line, the set  $K$  contains full lines.

**Lemma 4.9** *Every line intersecting  $K$  in  $c$  points, where  $2 \leq c < q^2 + 1$ , is contained in a medium plane.*

**Proof** Suppose a line  $L$  intersects  $K$  in  $c$  points, with  $2 \leq c < q^2 + 1$ , and has no medium planes through it. Consider a solid  $\Pi$  containing  $L$  and look at all planes through  $L$  inside  $\Pi$ . Since by assumption all these planes are large ones,  $\Pi$  would contain  $(q^2 + 1)(q^3 + q^2 + 1 - c) + c$  points belonging to the set  $K$ . This yields a contradiction, even for a non-singular solid, unless  $c = q^2 + 1$ , a case which is excluded.  $\square$

**Lemma 4.10** *If  $\alpha$  is a medium plane, it contains no full lines.*

**Proof** A medium plane  $\alpha$  is always contained in a non-singular solid  $\Pi$ . Suppose that  $L$  is a full line lying in  $\alpha$  and consider all planes through  $L$  inside  $\Pi$ . The proof of the foregoing lemma immediately shows that all these planes have to be non-singular ones, a contradiction.  $\square$

**Lemma 4.11** *If  $\alpha$  is a medium plane, then  $K_\alpha$  is a minimal blocking set in  $\alpha$ .*

**Proof** Consider a line  $L$  in  $\alpha$  containing  $c$  points of the set  $K$ , where  $2 \leq c < q^2 + 1$ . Note that such a line exists since a medium plane does not contain full lines by Lemma 4.10.

Consider a singular solid  $\Pi$  through  $\alpha$ , and look at all planes through  $L$  inside  $\Pi$ . We get at least

$$(q^3 + 1 - c)(q^2 + 1) + c$$

points belonging to  $\Pi \cap K$  because no small planes can contain  $L$  since all small planes intersect  $K$  in a line. This yields  $c \geq q$  otherwise  $\Pi$  would not be a singular solid. The case  $c = q$  cannot occur. Suppose the contrary. So take a line  $L$  such that  $L$  intersects  $K$  in  $q$  points. Consider a singular solid  $\Pi$  through  $L$ . The same counting as above yields that all planes through  $L$  inside  $\Pi$  must be medium planes. Hence,  $K$  would contain

$$(q^4 + q^2 + 1)(q^3 + 1 - q) + q = q^7 + q^4 + q^2 + 1$$

points, which yields a contradiction. So,  $c \geq q + 1$ . If  $K_\alpha$  would be a non-minimal blocking set, then there would be a point  $p$  of  $K_\alpha$  so that no line of  $\alpha$  through  $p$  is a tangent one. Hence, all lines through  $p$  would contain at least  $q + 1$  points. But then counting yields that  $K_\alpha$  contains at least  $1 + (q^2 + 1)q = q^3 + q + 1$  points, a contradiction.  $\square$

Using the foregoing lemma, one can easily determine the intersection numbers with lines.

**Lemma 4.12** *All lines intersect  $K$  in 1,  $q + 1$ , or  $q^2 + 1$  points.*

**Proof** Since every line intersecting  $K$  in  $c$  points, with  $2 \leq c < q^2 + 1$ , is contained in a medium plane and since medium planes are minimal non-trivial blocking sets, it follows from Theorem 3.2 that the blocking set is a unital and so  $c = q + 1$ .  $\square$

**Proof of Theorem 4.2.** A line  $L$  intersecting  $K$  in  $q + 1$  points is called a *Baer* line; a line intersecting  $K$  in  $q^2 + 1$  points is called a *full* line. For  $q > 2$  the proof is easily finished by Theorem 23.5.19 in [11]. So an additional argument is necessary for  $q = 2$ . However, the following reasoning works for all  $q$ . Consider a point  $p$  in  $K$  and a line  $L$  in  $K$  which are not incident. By Lemma 4.10 the plane generated by them is a large one. The lines through  $p$  in that plane all are Baer lines or full lines. Elementary counting yields exactly one of them is a full one.

Define  $S$  to be an incidence structure with as points the points of the set  $K$ , as lines the full lines, and where a point  $p$  and a line  $L$  are said to be incident if  $L$  passes through  $p$ . We have already proved that axioms (GQ2) and (GQ3) for generalized quadrangles hold for  $S$ . There is no point which is collinear with all others. If there was a point  $p$  collinear with all others, then consider a solid  $\Pi$  not incident with  $p$ . Since this solid contains either  $q^2(q^3 + 1) + 1$  or  $(q^2 + 1)(q^3 + 1)$  points, then

$$|K| \in \{1 + q^2(q^2(q^3 + 1) + 1), 1 + q^2((q^2 + 1)(q^3 + 1))\}$$

This yields a contradiction with the size of  $K$ . Next it will be shown that there is a point  $p$  with at least three lines through it. Let  $p$  be a point of  $K$  and let  $L$  be a full line of  $K$  with  $p$  not on  $L$ . As (GQ2) and (GQ3) are satisfied for the substructure  $S'$  induced by  $S$  in the plane  $\alpha$  through  $p$  and  $L$ , the geometry  $S'$  consists of  $q + 1$  lines in  $\alpha$  on  $p$ . So  $p$  is on at least 3 lines of  $S$ . Hence,  $S$  is a generalized quadrangle. Theorem 2.3 implies that it is a classical one and by looking at the different classical ones, it has to be a non-singular hermitian variety  $H(4, q^2)$ .  $\square$

## 5 Generalization

Consider a set of points  $K$  in  $PG(n, q^2)$ ,  $n \geq 4$ , for which each intersection number with planes and solids is also an intersection number with planes and solids of a hermitian variety. First recall what the intersections of hermitian varieties with planes and solids look like.

A plane lies on the hermitian variety, or intersects it in either a non-singular hermitian variety  $H(2, q^2)$ , a cone with vertex a point and base a non-singular hermitian variety  $H(1, q^2)$ , or a line.

A solid lies on the hermitian variety, or intersects it in either a non-singular hermitian variety  $H(3, q^2)$ , a cone with vertex a point and base a non-singular hermitian variety  $H(2, q^2)$ , a cone with vertex a line and base a non-singular hermitian variety  $H(1, q^2)$ , or a plane.

So the intersection numbers of the set  $K$  with planes belong to

$$q^2 + 1, q^3 + 1, q^3 + q^2 + 1, q^4 + q^2 + 1,$$

and the intersection numbers of the set  $K$  with solids belong to

$$q^4 + q^2 + 1, q^5 + q^2 + 1, q^5 + q^3 + q^2 + 1, q^5 + q^4 + q^2 + 1, q^6 + q^4 + q^2 + 1.$$

Call a plane, a solid or a line intersecting the set  $K$  in  $i$  points an *i-plane*, *i-solid* or *i-line*. A plane intersecting the set  $K$  in  $q^4 + q^2 + 1$  points will be called a *full plane*, a solid intersecting the set  $K$  in  $q^6 + q^4 + q^2 + 1$  points will be called a *full solid*, and a line intersecting the set  $K$  in  $q^2 + 1$  points will be called a *full line*.

**Lemma 5.1** *A  $(q^4 + q^2 + 1)$ -solid  $\Pi$  intersects the set  $K$  in a full plane.*

**Proof** Suppose some line  $L$  in  $\Pi$  does not intersect  $K$ . Consider all planes through  $L$  inside  $\Pi$ . Then at least  $(q^2 + 1)^2$  points are contained in  $|\Pi \cap K|$ , a contradiction. Hence all lines in  $\Pi$  are blocked by the set  $K$ . Hence, Theorem 3.4 implies that  $\Pi \cap K$  is a plane.  $\square$

**Lemma 5.2** *A  $(q^5 + q^3 + q^2 + 1)$ -solid  $\Pi$  does not contain full planes nor  $(q^2 + 1)$ -planes.*

**Proof** Let  $H1 = q^2 + 1$ ,  $H2 = q^3 + 1$ ,  $H3 = q^3 + q^2 + 1$ ,  $H4 = q^4 + q^2 + 1$  and  $x = q^5 + q^3 + q^2 + 1$ . Call the number of  $H1$ -,  $H2$ -,  $H3$ - and  $H4$ -planes inside  $\Pi$ ,  $a$ ,  $b$ ,  $c$  and  $t$  respectively. It will be shown that the parameter  $t$  has to be zero.

Then by counting the total number of planes in a solid, the incident pairs  $(p, \alpha)$ , where  $p$  is a point of  $K$  and  $\alpha$  a plane of  $\Pi$ , and the triples  $(p, r, \alpha)$  where  $p$  and  $r$  are distinct points of  $K$  and  $\alpha$  is a plane in  $\Pi$  containing  $p$  and  $r$ , in two ways, the following equations are obtained.

$$a + b + c = \frac{q^8 - 1}{q^2 - 1} - t,$$

$$aH1 + bH2 + cH3 = x \frac{q^6 - 1}{q^2 - 1} - tH4,$$

$$aH1(H1 - 1) + bH2(H2 - 1) + cH3(H3 - 1) = x(x - 1) \frac{q^4 - 1}{q^2 - 1} - tH4(H4 - 1).$$

Solving these equations yields  $a = -t(q^2 - q + 1)$ , hence  $t = 0$ .  $\square$

**Lemma 5.3** *A  $(q^5 + q^2 + 1)$ -solid  $\Pi$  contains at most  $q$  full planes. If it contains  $q$  full planes, then there are no  $(q^3 + q^2 + 1)$ -planes contained in  $\Pi$ . Furthermore, a  $(q^5 + q^2 + 1)$ -solid always contains  $(q^2 + 1)$ -planes.*

**Proof** Call the number of full planes inside  $\Pi$   $t$  as above. Using the same counting technique as above we find  $a = q^3 - tq^2 + tq - t + 1$ ,  $b = q^3(q^3 + t)$  and  $c = q^4 - (t + 1)q^3 + (1 + t)q^2 - tq$ . Since  $c$  has to be positive, it follows that  $t \leq q$ .  $\square$

**Lemma 5.4** *If a  $(q^5 + q^2 + 1)$ -solid  $\Pi$  does not contain lines intersecting  $K$  in  $q$  points then it does not contain full planes.*

**Proof** Suppose  $\Pi$  does contain a full plane  $\pi$ . By the proof of the previous lemma there exist  $(q^3 + 1)$ -planes in  $\Pi$ . Consider a  $(q^3 + 1)$ -plane  $\alpha$  in  $\Pi$  and a point  $p \in K \cap \alpha$  outside  $\pi$  and a line  $L$  in  $\Pi$  passing through  $p$ . If  $|L \cap K| = x$ , the following inequality holds.

$$(q^2 + 1)(q^3 + 1 - x) + x \leq q^5 + q^2 + 1.$$

Hence  $x \geq q$ . Since we assume there are no  $q$ -lines in  $\Pi$ , there follows that  $x \geq q + 1$ . Considering all lines through  $p$  inside  $\alpha$  yields a contradiction.  $\square$

**Lemma 5.5** *A  $(q^5 + q^4 + q^2 + 1)$ -solid contains  $q + 1$  full planes.*

**Proof** Apply exactly the same method as in the proofs of the lemmas above. Hence,  $a = q^3 + q^2 - q - 1 - t(q^2 - q + 1)$  and  $b = q^3(-q + t - 1)$ . The first equation implies  $t \leq q + 1$ , while the second implies  $t \geq q + 1$ , hence  $t = q + 1$ .  $\square$

**Lemma 5.6** *A  $(q^5 + q^4 + q^2 + 1)$ -solid  $\Pi$  is a union of  $q + 1$  full planes all passing through the same line  $L$ .*

**Proof** Suppose there does not pass a full plane lying in  $\Pi$  through  $p$ , with  $p \in K \cap \Pi$ . Consider an arbitrary line  $L$  in  $\Pi$  passing through  $p$  and intersecting  $K$  in  $x$  points. Consider all planes through  $L$  inside  $\Pi$ . Then at most

$$(q^2 + 1)(q^3 + q^2 + 1 - x) + x = q^5 + q^4 + q^2 + 1 + q^3 + (1 - x)q^2$$

points belong to  $\Pi \cap K$ . This implies  $x \leq q + 1$ . Consider all lines through  $p$  in a fixed plane  $\beta$  of  $\Pi$  through  $p$ ; they all contain at most  $q + 1$  points of  $K$ , hence  $\beta$  contains at most  $1 + (q^2 + 1)q = q^3 + q + 1$ , so at most  $q^3 + 1$  points of the set  $K$ . Hence all planes in  $\Pi$  through  $p$  would contain at most  $q^3 + 1$  points. Consider again an arbitrary line  $L$  in  $\Pi$  passing through  $p$  and repeat the argument above. We get the inequality

$$(q^2 + 1)(q^3 + 1 - x) + x \geq q^5 + q^4 + q^2 + 1.$$

This yields a contradiction. Hence through all points  $p \in \Pi \cap K$  there passes a full plane belonging to  $\Pi$ . Hence  $\Pi$  intersects  $K$  in a union of  $q + 1$  full planes. Since  $|\Pi \cap K| = q^5 + q^4 + q^2 + 1$  these planes all intersect in the same line, otherwise there would be fewer points contained in  $\Pi \cap K$ .  $\square$

**Lemma 5.7** *If a 4-space  $\Delta$  is such that  $\Delta \cap K = (q^2 + 1)(q^5 + 1)$  then  $\Delta \cap K$  is a non-singular hermitian variety  $H(4, q^2)$ .*

**Proof** Let  $S1 = q^4 + q^2 + 1$ ,  $S2 = q^5 + q^2 + 1$ ,  $S3 = q^5 + q^3 + q^2 + 1$ ,  $S4 = q^5 + q^4 + q^2 + 1$ ,  $S5 = q^6 + q^4 + q^2 + 1$  and  $x = q^7 + q^5 + q^2 + 1$ . Call the number of  $S1$ -,  $S2$ -,  $S3$ -,  $S4$ - and  $S5$ -solids inside  $\Pi$ ,  $a$ ,  $b$ ,  $c$ ,  $t_1$  and  $t_2$ , respectively. It will be shown that the parameters  $t_1$  and  $t_2$  have to be zero.

Then by counting the total number of solids in a 4-space, the incident pairs  $(p, \Pi)$ , where  $p$  is a point of  $K$  and  $\Pi$  a solid of  $\Delta$ , and the triples  $(p, r, \Pi)$  where  $p$  and  $r$  are distinct points of  $K$  and  $\Pi$  is a solid in  $\Delta$  containing  $p$  and  $r$ , in two ways, we obtain the following equations

$$a + b + c = \frac{q^{10} - 1}{q^2 - 1} - t_1 - t_2,$$

$$aS1 + bS2 + cS3 = x \frac{q^8 - 1}{q^2 - 1} - t_1 S4 - t_2 S5,$$

$$aS1(S1 - 1) + bS2(S2 - 1) + cS3(S3 - 1) = x(x - 1) \frac{q^6 - 1}{q^2 - 1} - t_1 S4(S4 - 1) - t_2 S5(S5 - 1).$$

Solving these equations yields  $a = -\frac{t_2(q^4 - q^3 + 2q^2 - q + 1) + t_1}{q^2 - q + 1}$ . Hence,  $t_1 = t_2 = a = 0$ . So the only solids that occur in  $\Delta$  are  $(q^5 + q^2 + 1)$ - and  $(q^5 + q^3 + q^2 + 1)$ -solids. If  $\Delta$  contains a full plane  $\pi$  then consider all solids in  $\Delta$  through  $\pi$ . We get at most

$$(q^2 + 1)(q^5 + q^3 - q^4) + q^4 + q^2 + 1$$

points, a contradiction. Hence the intersection numbers with planes and solids are those of Theorem 4.2, so  $\Delta \cap K$  is a non-singular hermitian variety  $H(4, q^2)$ .  $\square$

**Lemma 5.8** *If a 4-space  $\Delta$  contains a line  $L$  intersecting  $K$  in  $q$  points, then this line cannot be contained in a  $(q^2 + 1)$ -plane of  $\Delta$ . Furthermore, if  $L$  is only contained in  $(q^3 + 1)$ -planes of  $\Delta$ , then  $|K \cap \Delta| = q^7 + q^4 + q^2 + 1$ .*

**Proof** Clearly  $L$  cannot be contained in full planes of  $\Delta$ . First suppose that  $L$  is only contained in  $(q^3 + 1)$ -planes of  $\Delta$ . Then  $K \cap \Delta = (q^4 + q^2 + 1)(q^3 + 1 - q) + q = q^7 + q^4 + q^2 + 1$ . Next suppose that  $L$  is contained in a  $(q^2 + 1)$ -plane  $\alpha$  of  $\Delta$ . By Lemma 5.2  $\alpha$  is not contained in a  $(q^5 + q^3 + q^2 + 1)$ -solid of  $\Delta$ . Since  $(q^4 + q^2 + 1)$ - and  $(q^5 + q^4 + q^2 + 1)$ -solids do not contain lines intersecting  $K$  in  $q$  points all solids through  $\alpha$  in  $\Delta$  are  $(q^5 + q^2 + 1)$ -solids. Hence  $|\Delta \cap K| = (q^2 + 1)q^5 + (q^2 + 1)$ . This is impossible by the previous lemma.  $\square$

**Lemma 5.9** *In a 4-space  $\Delta$  a  $(q^3 + q^2 + 1)$ -plane  $\beta$  does not contain a  $q$ -line.*

**Proof** By the previous lemma a  $q$ -line can not be contained in a  $(q^2 + 1)$ -plane. If in some 4-space  $\Delta$ , a  $q$ -line  $L$  lies in a  $(q^5 + q^2 + 1)$ -solid  $\Pi$  then an easy counting learns that inside  $\Pi$ ,  $L$  is only contained in  $(q^3 + 1)$ -planes. Hence if a  $(q^3 + q^2 + 1)$ -plane  $\beta$  lying in  $\Delta$  contains  $L$ , then  $\beta$  is not contained in  $(q^5 + q^2 + 1)$ -solids of  $\Delta$ . Since  $(q^5 + q^4 + q^2 + 1)$ -solids do not contain  $q$ -lines, in this case

$$|K \cap \Delta| = (q^2 + 1)q^5 + q^3 + q^2 + 1 = q^7 + q^5 + q^3 + q^2 + 1.$$

Suppose  $\Delta$  contains a  $(q^2 + 1)$ -plane  $\alpha$  and look at all solids through  $\alpha$  inside  $\Delta$ . A  $(q^5 + q^3 + q^2 + 1)$ -solid and a full solid do not contain  $(q^2 + 1)$ -planes, hence

$$aq^4 + bq^5 + (q^2 + 1 - a - b)(q^5 + q^4) + q^2 + 1 = q^7 + q^5 + q^3 + q^2 + 1.$$

Solving this equation yields  $a = \frac{q^3 + (1-b)q - 1}{q^2}$ , but this is never an integer, contradiction. Hence  $\Delta$  does not contain  $(q^2 + 1)$ -planes. Since  $(q^4 + q^2 + 1)$ -,  $(q^5 + q^2 + 1)$ - and  $(q^5 + q^4 + q^2 + 1)$ -solids all contain  $(q^2 + 1)$ -planes, these solids do not occur in  $\Delta$ .

Consider a  $(q^5 + q^3 + q^2 + 1)$ -solid in  $\Delta$ . Such a solid contains  $(q^3 + 1)$ - and  $(q^3 + q^2 + 1)$ -planes. Consider all solids through a  $(q^3 + 1)$ -plane inside  $\Delta$ . By the previous all these solids are  $(q^5 + q^3 + q^2 + 1)$ -solids. Next, consider all solids through a  $(q^3 + q^2 + 1)$ -plane inside  $\Delta$ . Again these are all  $(q^5 + q^3 + q^2 + 1)$ -solids. This yields a contradiction since we get a different number of points in the respective cases.  $\square$

**Lemma 5.10** *A 4-space  $\Delta$  intersecting  $K$  in  $q^7 + q^4 + q^2 + 1$  points contains at most  $q$  full solids. Furthermore, if  $\Delta$  contains  $q$  full solids, then it contains no  $(q^5 + q^4 + q^2 + 1)$ -solids.*

**Proof** Let  $S1 = q^4 + q^2 + 1$ ,  $S2 = q^5 + q^2 + 1$ ,  $S3 = q^5 + q^3 + q^2 + 1$ ,  $S4 = q^5 + q^4 + q^2 + 1$ ,  $S5 = q^6 + q^4 + q^2 + 1$  and  $x = q^7 + q^4 + q^2 + 1$ . Call the number of  $S1$ -,  $S2$ -,  $S3$ -,  $S4$ - and  $S5$ -solids inside  $\Pi$ ,  $a$ ,  $b$ ,  $c$ ,  $t_1$  and  $t_2$ , respectively. The parameters  $t_1$  and  $t_2$  will be used in the calculations.

Then by counting the total number of solids in a 4-space, the incident pairs  $(p, \Pi)$ , where  $p$  is a point of  $K$  and  $\Pi$  a solid of  $\Delta$ , and the triples  $(p, r, \Pi)$  where  $p$  and  $r$  are distinct points of  $K$  and  $\Pi$  is a solid in  $\Delta$  containing  $p$  and  $r$ , in two ways, one obtains the following equations

$$a + b + c = \frac{q^{10} - 1}{q^2 - 1} - t_1 - t_2,$$

$$aS1 + bS2 + cS3 = x \frac{q^8 - 1}{q^2 - 1} - t_1 S4 - t_2 S5,$$

$$aS1(S1-1) + bS2(S2-1) + cS3(S3-1) = x(x-1) \frac{q^6 - 1}{q^2 - 1} - t_1 S4(S4-1) - t_2 S5(S5-1).$$

Solving this yields  $c = \frac{(q^4 - (q^3 - q^2)(1+t_2) - qt_2 - t_1)q^3}{q^2 - q + 1}$ . Since  $c$  must be positive, the proof is finished.  $\square$

**Lemma 5.11** *If all lines intersect  $K$  in 1,  $q$ ,  $q+1$  or  $q^2+1$  points inside a  $(q^3+1)$ -plane  $\alpha$  and if at least 2 lines in  $\alpha$  intersect  $K$  in  $q^2+1$  points, then  $\alpha$  intersects  $K$  in a cone with vertex a point and base a line intersecting in  $K$  in  $q$  points. If there is no line intersecting  $K$  in  $q$  points in a  $(q^3+1)$ -plane, then this plane intersects  $K$  in a unital.*

**Proof** Assume that  $\alpha$  contains more than one full line, so at least two, say  $L$  and  $M$ . These lines intersect in a point  $r$ . Take an arbitrary point  $p$  on  $L$  or  $M$  different from  $r$ . All lines through  $p$  inside  $\alpha$  different from  $L$  or  $M$  intersect  $K$  in  $q$  points otherwise one gets more than  $1 + q^2 + q^2(q-1) = q^3 + 1$  points. Consider now a point  $s$  in  $K \cap \alpha$  not lying on  $L$  or  $M$ . All lines through  $s$  inside  $\alpha$  not through  $r$  intersect  $K$  in  $q$  points. Hence the line  $rs$  is a full line. Hence, the point  $r$  is collinear with all other points, proving our claim. The second fact is well-known.  $\square$

**Lemma 5.12** *If a 4-space  $\Delta$  contains a line  $L$  intersecting  $K$  in  $q$  points, then it intersects  $K$  in  $q^7 + q^4 + q^2 + 1$  points. Furthermore all lines in  $\Delta$  intersect  $K$  in 1,  $q$ ,  $q+1$  or  $q^2+1$  points and  $(q^3 + q^2 + 1)$ -planes in  $\Delta$  do not contain  $q$ -lines.*

**Proof** By Lemma 5.8 and Lemma 5.9  $L$  is only contained in  $(q^3+1)$ -planes inside  $\Delta$ . Then Lemma 5.8 shows that  $|\Delta \cap K| = q^7 + q^4 + q^2 + 1$ . Consider all solids through a  $(q^3+1)$ -plane of  $\Delta$  inside  $\Delta$ . Then one gets at least

$$(q^2 + 1)(q^5 + q^2 - q^3) + q^3 + 1 = q^7 + q^4 + q^2 + 1$$

points. Hence a  $(q^3+1)$ -plane of  $\Delta$  is only contained in  $(q^5+q^2+1)$ -solids in  $\Delta$  and not contained in a  $(q^5+q^3+q^2+1)$ -solid in  $\Delta$ . This means there are no  $(q^5+q^3+q^2+1)$ -solids inside  $\Delta$ , since by Lemma 5.2 such solids contain  $(q^3+1)$ -planes.

Consider a  $(q^2+1)$ -plane in  $\Delta$ . If such a plane is not contained in a  $(q^4+q^2+1)$ -solid of  $\Delta$ , then at least  $(q^2+1)(q^5)+q^2+1$  points are contained in  $K \cap \Delta$ , a contradiction. Hence  $(q^2+1)$ -planes contained in  $\Delta$  intersect  $K$  in a line.



Consider a  $(q^3 + q^2 + 1)$ -plane  $\beta$  in  $\Delta$  and all solids through it inside  $\Delta$ . Then counting yields there passes exactly one  $(q^5 + q^4 + q^2 + 1)$ -solid through  $\beta$  in  $\Delta$ . Hence if a point  $p$  of  $K$  in  $\Delta$  does belong to a  $(q^3 + q^2 + 1)$ -plane of  $\Delta$ , then there is a full plane through  $p$  in  $\Delta$  by Lemma 5.6. Suppose there is a point  $p$  of  $K$  in  $\Delta$  such that there does not pass a full plane through  $p$  in  $\Delta$ . Then also no  $(q^3 + q^2 + 1)$ -plane passes through  $p$  in  $\Delta$ . Consider a line through  $p$  in  $\Delta$  that contains  $x$  points of  $K$ . This yields the following inequality

$$(q^4 + q^2 + 1)(q^3 + 1 - x) + x \geq q^7 + q^4 + q^2 + 1.$$

Hence  $x \leq q$ . Consider all lines through  $p$  inside a fixed plane  $\pi$  through  $p$  lying in  $\Delta$ . Then  $\pi$  contains at most  $(q^2 + 1)(q - 1) + 1$  points, hence  $\pi$  should be a  $(q^2 + 1)$ -plane, but such a plane contains a full line, a contradiction. Hence through all points  $p \in K \cap \Delta$ , there passes a full plane. Hence it follows that a  $(q^5 + q^2 + 1)$ -solid of  $\Delta$  is the union of full lines.

In a  $(q^4 + q^2 + 1)$ -, a  $(q^5 + q^4 + q^2 + 1)$ - and a full solid of  $\Delta$  all lines intersect  $K$  in 1,  $q + 1$  or  $q^2 + 1$  points. Hence if a line in  $\Delta$  does not intersect  $K$  in 1,  $q + 1$  or  $q^2 + 1$  points, then it lies in a  $(q^5 + q^2 + 1)$ -solid  $\Pi$  inside  $\Delta$ .

Consider a point  $p \in K \cap \Pi$ . Consider a line  $L$  through  $p$  in  $\Pi$  containing  $x$  points of  $K$ , where  $x$  is different from 1,  $q + 1$  and  $q^2 + 1$ . Consider all planes through  $L$  inside  $\Pi$ . Assume, by way of contradiction, that  $L$  is contained in a  $(q^3 + q^2 + 1)$ -plane of  $\Delta$ . As such a plane is contained in one  $(q^5 + q^4 + q^2 + 1)$ -solid of  $\Delta$ , we have  $x \in \{1, q + 1, q^2 + 1\}$  by Lemma 5.6, a contradiction. Hence, all planes of  $\Pi$  containing  $L$  are  $(q^3 + 1)$ -planes. This implies

$$(q^2 + 1)(q^3 + 1 - x) + x = q^5 + q^2 + 1.$$

Hence  $x = q$ , so all lines in  $\Delta$  intersect  $K$  in 1,  $q$ ,  $q + 1$  or  $q^2 + 1$  points.  $\square$

## 6 There is no line intersecting $K$ in $q$ points

In this section it is assumed that no line intersects the set  $K$  in  $q$  points. Define a point-line geometry  $S = (P, B, I)$ , where the points of  $P$  are the points of  $K$ , where the lines of  $B$  are the full lines and where incidence is containment.

**Theorem 6.1** *The geometry  $S$  is a Shult space.*

**Proof** Consider a point  $p$  of  $S$  and a line  $L$  of  $S$ , such that  $p$  and  $L$  are not incident. We prove the main axiom for the incidence relation of a Shult space, referring to it as the 1-or-all axiom; see page 2 for the definition of a Shult space.

Consider the plane  $\alpha$  generated by  $p$  and  $L$  and let  $\Delta$  be a 4-dimensional subspace containing  $\alpha$ . Several cases are distinguished.

By Lemma 5.4 all  $(q^5 + q^2 + 1)$ -solids in  $\Delta$  do not contain full planes.

1) Suppose there is a solid  $\Pi$  containing  $\alpha$  in  $\Delta$  that contains a full plane  $\beta$ . Then  $\Pi$  intersects  $K$  either in  $\beta$ , a  $(q^5 + q^4 + q^2 + 1)$ -solid, which is the union of  $q + 1$  full planes through a line  $M$ , or a full solid. In all cases the 1-or-all axiom holds.

2) Suppose  $\Delta$  does not contain full planes. Then  $\Delta$  also does not contain  $(q^4 + q^2 + 1)$ - or  $(q^5 + q^4 + q^2 + 1)$ -solids. By Theorem 4.2  $\Delta$  intersects  $K$  in a non-singular hermitian variety  $H(4, q^2)$ , hence the 1-or-all axiom holds.

3) Suppose  $\Delta$  contains a full plane  $\beta$ , but no solid containing  $\alpha$  inside  $\Delta$  contains a full plane. Consider a solid  $\Pi$  in  $\Delta$  containing  $\beta$ . If  $L$  belongs to  $\Pi$  then it is contained in a full plane lying in  $\Pi$  bringing us back to Case 1. So assume  $L$  and  $\Pi$  intersect in a point  $s$ . This point belongs to a full plane  $\gamma$  in  $\Pi$ , which might be equal to  $\beta$ . Consider the solid generated by  $p$  and  $\gamma$ . This solid is either a  $(q^5 + q^4 + q^2 + 1)$ -solid or a full solid. Inside this solid there is a full plane  $\psi$  that passes through  $p$ . If  $L$  intersects  $\psi$  we are back in Case 1, so assume  $L \cap \psi = \emptyset$ . Consider an arbitrary point  $r \in L$  and the solid  $\Pi_r$  generated by  $r$  and  $\psi$ . This solid intersects  $K$  in a full solid or in a cone with vertex a line and base a line intersecting  $K$  in  $q + 1$  points. Hence the line  $rp$  intersects  $K$  in  $q + 1$  or  $q^2 + 1$  points. Consider all lines through  $p$  inside  $\alpha$ ; then the assumptions on the intersection sizes of planes with the set  $K$  imply that the 1-or-all axiom holds.  $\square$

**Theorem 6.2** *If  $S$  is non-degenerate, then it is a non-singular hermitian variety in  $PG(n, q^2)$ ,  $n \geq 4$ .*

**Proof** If there exists a full plane, then  $S$  is a non-degenerate Shult space of finite rank at least 3, and since all lines contain at least three points by definition,  $S$  with all its subspaces is a polar space. By Theorem 2.1 it is a finite classical polar space and by looking at the intersection numbers,  $S$  is a non-singular hermitian variety.

If there exists no full plane, then the previous arguments show we have proved that axiom (GQ3) for generalized quadrangles is satisfied for  $S$ . Clearly, there is a point  $p$  through which there pass three lines of  $S$ . Hence,  $S$  is a generalized quadrangle.

By Theorem 2.3, it is a classical one; going through the list of classical generalized quadrangles yields it is the non-singular hermitian variety  $H(4, q^2)$ .  $\square$

Suppose now that  $S$  is degenerate, so there exist points collinear with all other points. Such points are called singular points.

**Lemma 6.3** *The singular points of  $S$  form a subspace  $\Pi_k$  of  $PG(n, q^2)$ .*

**Proof** Take two singular points  $p$  and  $r$  of  $S$  and consider a point  $t$  lying on the line  $L = pr$ . We claim that  $t$  is a singular point. Since  $p$  and  $r$  are both singular, all the points of  $L$  belong to  $S$ , so  $t \in S$ . Let  $M \neq L$  be an arbitrary line through  $t$  and consider the plane  $\alpha$  generated by  $L$  and  $M$ . If  $\alpha$  intersects  $S$  in  $L$  then  $M$  intersects  $S$  in 1 point. If  $\alpha$  contains a point  $n$  of  $S$  not belonging to  $L$ , then through  $n$  there pass at least two full lines in  $\alpha$  since  $p$  and  $r$  are singular points. Since we have the 1-or-all axiom all lines through  $n$  in  $\alpha$  are full lines. Hence  $\alpha$  is a full plane and  $M$  is a line of  $S$ . So every line through  $t$  intersects  $S$  either in 1 or in  $q^2 + 1$  points.  $\square$

**Lemma 6.4** *If  $S$  contains singular points, then all lines not intersecting the subspace  $\Pi_k$  formed by the singular points, intersect  $S$  in 1,  $q + 1$ , or  $q^2 + 1$  points.*

**Proof** Consider a line  $L$  not intersecting  $\Pi_k$ . Take a singular point  $p$  and consider the plane generated by  $p$  and  $L$ . Since this plane contains either  $q^2 + 1$ ,  $q^3 + 1$ ,  $q^3 + q^2 + 1$  or  $q^4 + q^2 + 1$  points of  $S$  by assumption, the lemma is proved.  $\square$

**Lemma 6.5** *If  $n - k - 1 \geq 4$ , then  $S$  is a cone with vertex a  $k$ -dimensional space and base a non-singular hermitian variety.*

**Proof** If  $S$  is degenerate, then look at a complementary space  $PG(n - k - 1, q^2)$  of the space  $\Pi_k$ . By assumption, this space does not contain singular points of  $S$ . If  $n - k - 1 \geq 4$ , then Theorem 6.2 shows that  $S$  intersects this space in a non-singular hermitian variety, hence  $S$  is a cone with vertex a  $k$ -dimensional space and base a non-singular hermitian variety.  $\square$

Now, consider all other cases one by one.

- (a) If  $n - k - 1 = -1$ , then  $S$  is the projective space  $PG(n, q^2)$ .
- (b) If  $n - k - 1 = 0$ , then  $S$  is a hyperplane of  $PG(n, q^2)$ .
- (c) If  $n - k - 1 = 1$ , then the complementary space is a line. If this line intersects  $K$  in one or in  $q^2 + 1$  points, a contradiction. If it intersects  $K$  in  $q + 1$  points, it is the union of  $q + 1$  hyperplanes.
- (d) If  $n - k - 1 = 2$ , then the complementary space is a plane  $\pi$ . By Lemma 6.4 all lines intersect in 1,  $q + 1$  or  $q^2 + 1$  points.
  - (1) Suppose that  $\pi$  intersects  $S$  in  $q^2 + 1$  points. Since all lines are blocked, this intersection has to be a line, but then there are singular points in the base, a contradiction.
  - (2) Suppose that  $\pi$  intersects  $S$  in  $q^3 + 1$  points. Since we assume no lines intersect  $K$  in  $q$  points, all lines in  $\pi$  intersect  $S$  in 1 or in  $q + 1$  points, hence  $\pi$  intersects  $S$  in a unital. Indeed, if there would be a full line  $L$  in  $\pi$  then consider a point  $p$  in  $\pi \cap S$  not belonging to  $L$ , and consider all lines through  $p$  in  $\pi$ . At least  $1 + (q^2 + 1)q = q^3 + q + 1$  points would be contained in  $\pi \cap K$ , a contradiction.
  - (3) Suppose that  $\pi$  intersects  $S$  in  $q^3 + q^2 + 1$  points. Since there is no line in  $\pi$  that intersects  $K$  in  $q$  points, the 1-axiom for generalized quadrangles is fulfilled. Hence some point  $s \in S \cap \pi$  is collinear with all other points in  $S \cap \pi$  otherwise we have a generalized quadrangle fully embedded in  $\pi$ , which yields a contradiction. But then  $s$  is a singular point, which is impossible.
- (e) If  $n - k - 1 = 3$ , then the complementary space is a solid  $\Pi$ .

By assumption, the solid  $\Pi$  does not contain a line intersecting  $S$  in  $q$  points. If  $\Pi$  contains a full plane then by Lemmas 5.2 and 5.4 it is either a  $(q^4 + q^2 + 1)$ -solid, a  $(q^5 + q^4 + q^2 + 1)$ -solid or a full solid. In all cases there are singular points in the base, a contradiction. Hence  $\Pi$  does not contain full planes. Then Lemma 4.1 implies that  $\Pi \cap K = H(3, q^2)$ .

**Theorem 6.6** *If a set of points  $K$  in  $PG(n, q^2)$ ,  $n \geq 4$ , contains no lines intersecting  $K$  in  $q$  points, and if the intersection numbers of  $K$  with planes and solids are also intersection numbers of planes and solids with a hermitian variety in  $PG(n, q^2)$ , then  $K$  is either*

- (i) *The projective space  $PG(n, q^2)$ ,*
- (ii) *A hyperplane in  $PG(n, q^2)$ ,*
- (iii) *A hermitian variety in  $PG(n, q^2)$ .*
- (iv) *A cone with vertex an  $(n - 2)$ -dimensional space and base a line intersecting  $K$  in  $q + 1$  points.*
- (v) *A cone with vertex an  $(n - 3)$ -dimensional space and base a unital.*

## 7 There is a line intersecting $K$ in $q$ points

In this section it is assumed that there is a line intersecting the set  $K$  in  $q$  points. By Lemma 5.12, every line intersects the set  $K$  either in 1,  $q$ ,  $q + 1$  or  $q^2 + 1$  points, and a line intersecting  $K$  in  $q$  points is only contained in  $(q^3 + 1)$ -planes. The following notations are introduced:

$$\begin{aligned} H_1(k) &= \frac{q^{2k} - 1}{q^2 - 1}, \\ H_2(k) &= q^{2k-1} + \frac{q^{2k-2} - 1}{q^2 - 1}, \\ H_3(k) &= q^{2k-1} + \frac{q^{2k} - 1}{q^2 - 1}, \\ H_4(k) &= \frac{q^{2k+2} - 1}{q^2 - 1}. \end{aligned}$$

**Lemma 7.1** *Assume  $K$  is contained in  $\Pi_n = PG(n, q^2)$ .*

- (i) *Then  $|K| = H_2(n)$ .*
- (ii) *Next, consider any subspace  $\Pi_l$  of dimension  $l$  in  $\Pi_n$ , with  $l \geq 1$ . Then  $|\Pi_l \cap K| \in \{H_1(l), H_2(l), H_3(l), H_4(l)\}$ .*

**Proof** (i) Since a  $q$ -line of  $\Pi_n$  is only contained in  $(q^3 + 1)$ -planes, there holds

$$|\Pi_n \cap K| = (q^3 + 1 - q) \frac{q^{2n-2} - 1}{q^2 - 1} + q = H_2(n).$$

(ii) The theorem clearly holds for  $l = 1$  and  $l = 2$ . Now assume that  $l \geq 3$ . If  $\Pi_l$  contains a  $q$ -line, then  $|\Pi_l \cap K| = H_2(l)$ . So assume that  $\Pi_l$  does not contain a  $q$ -line. If

$M$  is a full line in  $\Pi_l$  and  $p$  is a point of  $K$  in  $\Pi_l$  not on  $M$ , then the plane  $\alpha$  generated by  $M$  and  $p$  is not a  $(q^3 + 1)$ -plane by Lemma 5.11. If  $\alpha$  is a  $(q^3 + q^2 + 1)$ -plane, then an easy counting argument shows that  $\alpha$  contains exactly one full line containing  $p$ . So the 1-or-all axiom holds in  $\Pi_l$ . First, let  $l = 3$ . If  $\pi$  is a  $(q^2 + 1)$ -plane in  $\Pi_3$ , then  $\pi \cap K$  is a full line, and  $\Pi_3 \cap K$  either is a plane, a cone with as vertex a point  $p$  and as base a unital, or a set of  $q + 1$  planes containing a common line. If  $\Pi$  does not contain a  $(q^2 + 1)$ -plane, but does contain a full plane, then  $\Pi_3 \cap K = \Pi_3$ . If  $\Pi_3$  does not contain a  $(q^2 + 1)$ -plane, nor a full plane, then by Theorem 4.1  $\Pi_3 \cap K = H(3, q^2)$ . In each of these three cases we have  $|\Pi_3 \cap K| \in \{H_1(3), H_2(3), H_3(3), H_4(3)\}$ .

Next, let  $l \geq 4$ . Then by Theorem 6.6,  $|\Pi_l \cap K| \in \{H_1(l), H_2(l), H_3(l), H_4(l)\}$ .  $\square$

**Lemma 7.2** (i) *If  $|K \cap \Pi_l| = H_1(l)$ ,  $l \geq 1$ , then  $K \cap \Pi_l$  is an  $(l - 1)$ -dimensional space.*

(ii) *If  $|K \cap \Pi_l| = H_3(l)$ ,  $l \geq 1$ , then  $K \cap \Pi_l$  is a union of  $q + 1$   $(l - 1)$ -dimensional spaces containing a common space  $\Pi_{l-2}$  of dimension  $l - 2$ .*

**Proof** (i) Since all lines of  $\Pi_l$  are blocked by  $K$  and  $|K \cap \Pi_l| = \frac{q^{2l}-1}{q^2-1}$ , the result follows from Theorem 3.4.

(ii) Follows easily from the proof of Lemma 7.1.  $\square$

**Lemma 7.3** *Inside a space  $\Pi_l$  intersecting  $K$  in  $H_2(l)$  points, an  $(l - 2)$ -dimensional space  $\Pi_{l-2}$  intersecting  $K$  in  $H_3(l - 2)$  points is contained in exactly one  $(l - 1)$ -dimensional space intersecting  $K$  in  $H_3(l - 1)$  points.*

**Proof** By Lemma 7.2,  $\Pi_{l-2}$  can only be contained in  $(l - 1)$ -dimensional spaces of  $\Pi_l$  intersecting  $K$  in  $H_2(l - 1)$  or  $H_3(l - 1)$  points. If  $x$  of them intersect in  $H_3(l - 1)$  points, one gets the following equation

$$(q^2 + 1 - x)(H_2(l - 1) - H_3(l - 2)) + x(H_3(l - 1) - H_3(l - 2)) + H_3(l - 2) = H_2(l).$$

Solving yields  $x = 1$ .  $\square$

**Lemma 7.4** *For every space  $\Pi_l$ ,  $l \geq 2$ , the set  $K \cap \Pi_l$  is a union of  $(l - 2)$ -dimensional spaces.*

**Proof** The result clearly holds for  $l = 2$ . So assume that  $l \geq 3$ . If  $K \cap \Pi_l$  does not contain a line intersecting  $K$  in  $q$  points, the result is clear by Lemma 7.2 and the proof of Lemma 7.1.

So assume  $K \cap \Pi_l$  does contain such a line. Consider a point  $p \in K \cap \Pi_l$  and suppose  $p$  is not contained in an  $(l - 2)$ -dimensional space of  $\Pi_l \cap K$ . Let  $\Pi_{l-1}$  be an  $(l - 1)$ -dimensional subspace of  $\Pi_l$  containing  $p$ . By induction  $\Pi_{l-1}$  contains an  $(l - 3)$ -dimensional space  $\Pi_{l-3}$  of  $\Pi_{l-1} \cap K$  containing  $p$ . By Lemma 7.2 necessarily  $|K \cap \Pi_{l-1}| = H_2(l - 1)$ . Let  $\Pi_{l-2}$  be an  $(l - 2)$ -dimensional subspace of  $\Pi_l$  containing  $\Pi_{l-3}$ .

Counting points in the spaces  $\Pi_{l-1} \subset \Pi_l$  containing  $\Pi_{l-2}$  and putting  $|\Pi_{l-2} \cap K| = x$ , yields

$$(q^2 + 1)(H_2(l - 1) - x) + x = H_2(l).$$

So  $x = H_2(l - 2)$ . Counting points in the spaces  $\Pi_{l-2} \subset \Pi_l$  containing  $\Pi_{l-3}$  and putting  $|\Pi_{l-3} \cap K| = y$ , gives us

$$(q^4 + q^2 + 1)(H_2(l - 2) - y) + y = H_2(l).$$

So  $y = H_2(l - 3)$ . This contradicts  $\Pi_{l-3} \subset K$ .  $\square$

**Lemma 7.5** *The set  $K$  is a cone with vertex an  $(n - 3)$ -dimensional space and base a  $(q^3 + 1)$ -plane.*

**Proof** (1) Suppose  $K$  does not contain a hyperplane. There are only three types of hyperplane intersections by Lemma 7.1. The arguments sketched below can be worked out in detail exactly as in Lemma 5.10. One can count how many hyperplane intersections of each type there are. Furthermore, if a hyperplane intersects  $K$  in  $H_3(n - 1)$  points, one can count how many  $(n - 2)$ -dimensional spaces inside the hyperplane intersect  $K$  in  $H_3(n - 2)$  points. By Lemma 7.3 above each  $(n - 2)$ -dimensional space intersecting  $K$  in  $H_3(n - 2)$  points is contained in exactly one  $(n - 1)$ -dimensional space intersecting  $K$  in  $H_3(n - 1)$  points. In this way one can count the number of  $(n - 2)$ -dimensional spaces intersecting  $K$  in  $H_3(n - 2)$  points.

The number of  $(n - 2)$ -dimensional spaces intersecting  $K$  in  $H_3(n - 2)$  points can also be counted directly in function of the number of  $(n - 2)$ -dimensional spaces contained in  $K \cap \Pi_n$ . Comparing the two expressions yields a linear equation for the number of  $(n - 2)$ -dimensional spaces contained in  $K \cap \Pi_n$ . So there can only be one solution. Since there is the example of a cone with vertex an  $(n - 3)$ -dimensional space and base a unital the solution is  $q^3 + 1$ .

Furthermore, a standard counting argument yields there are also exactly  $q^3 + 1$  hyperplanes that intersect  $K$  exactly in an  $(n - 2)$ -dimensional space.

By Lemma 7.4  $\Pi_n \cap K$  also has to be the union of the  $q^3 + 1$   $(n - 2)$ -dimensional spaces found above.

They all have to contain a common  $(n - 3)$ -dimensional space. Counting the points of  $K$  in the  $q^2 + 1$  hyperplanes containing a given  $(n - 2)$ -dimensional space  $\Pi_{n-2}$  in  $K$ , implies that  $\Pi_{n-2}$  is contained in exactly one hyperplane  $\Pi_{n-1}$  for which  $\Pi_{n-1} \cap K = \Pi_{n-2}$ , and in no hyperplanes containing  $H_2(n - 1)$  points of  $K$ . Let  $\Pi'_{n-2}$  be a second  $(n - 2)$ -dimensional space of  $K$ . Then  $\Pi'_{n-2} \cap \Pi_{n-1}$  is an  $(n - 3)$ -dimensional space contained in  $\Pi_{n-2}$ . So every two  $(n - 2)$ -dimensional spaces completely contained in  $K$  intersect each other in an  $(n - 3)$ -dimensional space. Dualizing, one sees that all  $(n - 2)$ -dimensional spaces contained in  $K$  either contain a common  $(n - 3)$ -dimensional space, or are contained in a common hyperplane. Clearly, the latter cannot occur.

(2) Suppose  $K$  does contain a hyperplane  $\Pi_{n-1}$ . We know  $K$  is the union of  $(n - 2)$ -dimensional spaces. Take a point  $p \notin \Pi_{n-1}$  and consider the  $(n - 2)$ -dimensional space

through  $p$  contained in  $K$ . This space intersects  $\Pi_{n-1}$  in an  $(n-3)$ -dimensional space  $\Pi_{n-3}$ . Take an arbitrary point  $r \in \Pi_{n-3}$ , a point  $s \in K \setminus (\Pi_{n-1} \cup \Pi_{n-2})$ , and a point  $t$  in  $\Pi_{n-2} \setminus \Pi_{n-3}$ . Consider the plane  $\pi_{rst}$  generated by  $r$ ,  $s$  and  $t$ . Due to Lemma 5.11 and Lemma 7.2 the line  $rs$  is a full line. This finishes the proof.  $\square$

Together with the results of Theorem 6.6 and Lemma 5.11 we have proved our main result.

**Theorem 7.6** *If for a set of points  $K$  in  $PG(n, q^2)$ ,  $n \geq 4$  the intersection numbers with planes and solids are also intersection numbers of planes and solids with a hermitian variety, then  $K$  is either*

- (i) *The projective space  $PG(n, q^2)$ ,*
- (ii) *A hyperplane in  $PG(n, q^2)$ ,*
- (iii) *A hermitian variety in  $PG(n, q^2)$ ,*
- (iv) *A cone with vertex an  $(n-2)$ -dimensional space and base a line intersecting  $K$  in  $q$  or  $q+1$  points,*
- (v) *A cone with vertex an  $(n-3)$ -dimensional space and base a unital,*
- (vi) *A cone with vertex an  $(n-3)$ -dimensional space and base a set  $\tilde{K}$  of  $PG(2, q^2)$  intersecting each line of  $PG(2, q^2)$  in 1,  $q$ ,  $q+1$  or  $q^2+1$  points and containing exactly one full line.*

**Remark.** Let  $\mathcal{M}$  be a maximal  $\{q^3 - q^2 + q; q\}$ -arc in  $PG(2, q^2)$ , that is, a point set  $\mathcal{M}$  of size  $q^3 - q^2 + q$  in  $PG(2, q^2)$  intersecting each line of  $PG(2, q^2)$  in either 0 or  $q$  points; such a set is known to exist for any  $q = 2^h$ , see [10]. Let  $M$  be a line of  $PG(2, q^2)$  intersecting  $\mathcal{M}$  in  $q$  points and let  $L$  be a line of  $PG(2, q^2)$  containing no points of  $\mathcal{M}$ . Then  $\tilde{K} = (\mathcal{M} \setminus M) \cup L$  can be taken as base of the cone described in Theorem 7.6 (vi).

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## References

- [1] Simeon Ball. Multiple blocking sets and arcs in finite planes. *J. London Math. Soc. (2)*, 54(3):581–593, 1996.
- [2] A. Blokhuis, L. Storme, and T. Szőnyi. Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)*, 60(2):321–332, 1999.

- [3] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combin. Theory*, 1:96–104, 1966.
- [4] A. Bruen. Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.*, 76:342–344, 1970.
- [5] A. A. Bruen. Polynomial multiplicities over finite fields and intersection sets. *J. Combin. Theory Ser. A*, 60(1):19–33, 1992.
- [6] A. A. Bruen and J. A. Thas. Blocking sets. *Geometriae Dedicata*, 6(2):193–203, 1977.
- [7] F. Buekenhout and C. Lefèvre. Generalized quadrangles in projective spaces. *Arch. Math. (Basel)*, 25:540–552, 1974.
- [8] F. Buekenhout and E.E. Shult. On the foundations of polar geometry. *Geom. Dedicata*, 3:155–170, 1974.
- [9] O. Ferri and G. Tallini. A characterization of nonsingular quadrics in  $\text{PG}(4, q)$ . *Rend. Mat. Appl. (7)*, 11(1):15–21, 1991.
- [10] J. W. P. Hirschfeld. *Projective geometries over finite fields, Second edition*. Oxford University Press, Oxford, 1998, 555pp.
- [11] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Oxford University Press, Oxford, 1991, 407pp.
- [12] S. E. Payne and J. A. Thas. *Finite generalized quadrangles*. Pitman, London, 1984, 312pp.
- [13] J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386, Springer, Berlin, 1974, 299pp.
- [14] F.D. Veldkamp. Polar geometry, i-v. *Indag. Math.*, 21:512–551, 1959.