A FLAT LAGUERRE PLANE OF KLEINEWILLINGHÖFER TYPE V

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(May 5, 2011)

Abstract

Kleinewillinghöfer types of Laguerre planes reflect transitivity properties of certain groups of central automorphisms. In the case of flat Laguerre planes, Polster and Steinke have shown that some of the conceivable types cannot exist, and they gave models for most of the other types. Only few types are still in doubt. One of them is type V.A.1, whose existence we prove here. In particular, this settles all questions about existence of types defined by Laguerre homologies. In order to construct our model, we make systematic use of the restrictions imposed by the group. We conjecture that our example belongs to a one-parameter family of planes, all of type V.A.1.
MSC 2000: 51H15, 51B15.

Keywords and phrases: Laguerre plane, Kleinewillinghöfer type, topological geometry.

1. Introduction

Kleinewillinghöfer [5] classified Laguerre planes with respect to linearly transitive groups of central automorphisms, see Section 3 for definitions. In [11] and [16] flat Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types of their full automorphism groups. In particular, all possible types of flat Laguerre planes with respect to Laguerre translations were completely determined in [11], and the case of Laguerre homotheties was dealt with in [16]. Examples for some of the possible Kleinewillinghöfer types of flat Laguerre planes can be found in [11, Section 6], [8] and [17].

In this paper we provide an example for a flat Laguerre plane of Kleinewillinghöfer type V.A.1. With this model all Kleinewillinghöfer types with respect to Laguerre homologies are now accounted for, and the number of open cases of combined Kleinewillinghöfer types is reduced to two. Moreover, numerical evidence suggests that this example belongs to an infinite one-parameter family of flat Laguerre planes, all of Kleinewillinghöfer type V.A.1.

The authors wish to thank the referees for their careful reading of the manuscript and their valuable comments and corrections.

2. Flat Laguerre Planes

A *flat Laguerre plane* (or *topological, locally compact, 2-dimensional Laguerre plane*) $\mathcal{L} = (Z, \mathcal{C})$ is an incidence structure of points and circles whose point set is the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$, whose circles $C \in \mathcal{C}$ are graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$ such that any three points no two of which are on the same generator $\{c\} \times \mathbb{R}$ of the cylinder can be joined by a unique circle and such that the circles which touch a fixed circle K at $p \in K$ partition the complement in Z of the generator that contains p. For more information on flat Laguerre planes we refer to [2] and [3] or [10, Chapter 5]. (There are also topological, locally compact 0- and 4-dimensional Laguerre planes.) The generators of Z are usually referred to as the parallel classes of $\mathcal{L}$. We denote the parallel class of a point p by $|p|$. We further say that two points a parallel if they belong to the same parallel class.

For each point p of $\mathcal{L}$ we form the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set A_p consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p. It readily follows that $\mathcal{A}_p$ is an affine plane. We call $\mathcal{A}_p$ the *derived affine at* p.

Each derived affine plane $\mathcal{A}_p$ of a flat Laguerre plane is even a topological, locally compact, 2-dimensional affine plane and extends to a topological, compact, 2-dimensional projective plane $\mathcal{P}_p$, which we call the *derived projective plane at* p. Circles not passing through the distinguished point p induce closed ovals in $\mathcal{P}_p$ by removing the point parallel to p and adding in $\mathcal{P}_p$ the point ω at infinity of the lines that come from parallel classes of $\mathcal{L}$. The line at infinity of $\mathcal{P}_p$ (relative to $\mathcal{A}_p$) is a tangent to this oval.

The *classical real Laguerre plane* is obtained as the geometry of non-trivial plane sections of a cylinder in $\mathbb{R}^3$ with an ellipse in $\mathbb{R}^2$ as base, or equivalently, as the geometry of non-trivial plane sections of an elliptic cone, in real 3-dimensional projective space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in the construction of the classical flat Laguerre plane by arbitrary closed ovals in $\mathbb{R}^2$, i.e., convex, differentiable simply closed curves, we also obtain flat Laguerre-planes. These are the so-called *flat ovoidal Laguerre planes*.

An automorphism of a Laguerre plane is a permutation of its point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. Every automorphism of a flat Laguerre plane is continuous and thus a homeomorphism of Z. The collection of all automorphisms of a flat Laguerre plane $\mathcal{L}$ forms a group with respect to composition, the automorphism group Γ of $\mathcal{L}$. This group is a Lie group with respect to the compact-open topology; see [14]. In general, Γ is not connected. We call the dimension of Γ the *group dimension* of $\mathcal{L}$. (Note that $\dim \Gamma = \dim \Gamma^1$ where Γ^1 is the connected component of Γ that contains the identity element.) The maximum group dimension is 7, and it is attained precisely in the classical real Laguerre plane. Group dimension 6 does not occur. Furthermore, flat Laguerre planes of group dimension 5 must be special ovoidal Laguerre planes; see [7, Theorem 1].

3. Kleinewillinghöfer types of flat Laguerre planes

Kleinewillinghöfer considered four kinds of central automorphisms of Laguerre planes: C-homologies, G-translations, $(G, B(q, C))$-translations and (p, q)-homotheties; see the following for definitions. *Central automorphisms* are automorphisms that have at least one fixed point and induce central collineations in the derived projective plane at this fixed point. The four different kinds of central automorphisms above are distinguished according to the relative position of centre and axis and whether or not the axis is the line at infinity of the derived affine plane at one of its fixed points. The notions of translation, homothety and homology describe the sort of central collineation one sees in this derived affine plane.

A subgroup of central automorphisms that have the same 'centre' and 'axis' is *linearly transitive* if the induced group of central collineations in a derived projective plane at one of the fixed points is transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. Kleinewillinghöfer considered groups of automorphisms and determined their types according to linearly transitive subgroups of central automorphisms contained in them. A group of automorphisms is said to be linearly transitive if it contains a linearly transitive subgroup of central automorphisms. The Kleinewillinghöfer type of a Laguerre plane is defined to be the type of its full automorphism group.

A *Laguerre homology* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one circle. One speaks of a *C-homology* if C is the circle that is fixed pointwise. For each point $q \in C$, a C-homology induces a homology of the derived projective plane $\mathcal{P}_q$. The centre of the induced homology is the point ω at infinity, corresponding to the parallel classes of $\mathcal{L}$. With respect to Laguerre homologies Kleinewillinghöfer obtained seven types of groups of automorphisms of Laguerre planes, labelled I, II, III, IV, V, VI and VII; see [5, Satz 3.1]. Of these types type VI cannot occur as the type of a flat Laguerre plane; see

[11, Proposition 3.4]. With the examples from [11, Section 6], [8], [17], [13] and the model given in the present paper, all remaining types with respect to Laguerre homologies are known occur.

A *Laguerre translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one parallel class and induces a translation in the derived affine plane at one of its fixed points. Laguerre translations come in two different varieties. Firstly, a non-identity *G-translation* of $\mathcal{L}$ is a Laguerre translation that fixes precisely the points of the parallel class G and, furthermore, fixes each parallel class globally. For the second variety of Laguerre translations we consider a tangent bundle $B(p, C)$, that is, all circles that touch the circle C at the point p. In the derived affine plane at p the tangent bundle represents a bundle of parallel lines, and we can look at translations in this direction. Then a *$(G, B(p, C))$-translation* of $\mathcal{L}$ is a Laguerre translation that fixes C (and each circle in $B(p, C)$) globally. With respect to Laguerre translations Kleinewillinghöfer obtained 11 types of groups of automorphisms of Laguerre planes, labelled A through to K; see [5, Satz 3.3], or [6, Satz 2]. Of these types the types F, I and J cannot occur as types of flat Laguerre planes; see [11, Proposition 4.8]. There are examples for flat Laguerre planes for each of the remaining types with respect to Laguerre translations; see [11, Section 6].

Finally, a *Laguerre homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely two non-parallel points and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a *$\{p, q\}$-homothety* if p, q are the two fixed points. With respect to Laguerre homotheties Kleinewillinghöfer [5, Satz 3.2] or [6, Satz 1], obtained 13 types of groups of automorphisms of Laguerre planes, labelled 1 through to 13. Types 5, 6, 7, 9, 10 and 12 cannot occur as types of flat Laguerre planes; see [11, Proposition 5.6] and [16]. There are examples for flat Laguerre planes for each of the remaining types with respect to Laguerre homotheties; see [11, Section 6] and [17].

Combining all three classifications Kleinewillinghöfer obtained a total of 46 combined types. In flat Laguerre planes 21 of these 46 types cannot occur. There are models of flat Laguerre planes of types I.A.1, I.B.1, I.B.3, I.C.1, I.E.1, I.E.4, I.G.1, I.H.1, I.H.11, II.A.1, II.E.1, II.E.4, II.G.1, III.B.1, III.B.3, III.H.1, III.H.11, IV.A.1, IV.A.2, VII.D.1, VII.D.8 and VII.K.13; see [11, Section 6], [8], [15], [16], [17], [13]. Here a combined type just refers to the respective simple types. Note that, with the model given in this paper, there is a flat Laguerre plane of each of the simple Kleinewillinghöfer types not excluded in [11] and [17]. Furthermore, with the examples from [11, Section 6], [8], [17], [13] and the model given in this paper, only the existence of combined types I.A.2 and II.A.2 remains open in flat Laguerre planes.

4. The general setting for a flat Laguerre plane of Kleinewillinghöfer type V

In this section, we consider a Laguerre plane $\mathcal{L}$ of type V.A.1. This means that the set $\mathcal{Z}$ of all circles C for which the automorphism group of $\mathcal{L}$ is linearly transitive (with respect to C-homologies) consists of a flock of $\mathcal{L}$, that is, the circles in $\mathcal{Z}$ partition the point set of $\mathcal{L}$ (type V), that there is neither a tangent bundle nor a parallel class for which the group of Laguerre translations is linearly transitive (type A), and that there is no group of Laguerre homotheties that is linearly transitive (type 1).

THEOREM 4.1. *A flat Laguerre plane $\mathcal{L}$ of Kleinewillinghöfer type V with respect to Laguerre homologies has group dimension 2 or 3. In the latter case, the automorphism group of $\mathcal{L}$ has precisely two orbits on the circle space $\mathcal{C}$. One orbit consists of all the circles in the flock $\mathcal{F}$ as in type V, and the other orbit is $\mathcal{C} \setminus \mathcal{F}$.*

PROOF. Let $\mathcal{L}$ be a flat Laguerre plane of Kleinewillinghöfer type V and let $\mathcal{F}$ be the flock of $\mathcal{L}$ as in type V. Let G be the group generated by all Laguerre homologies at circles in $\mathcal{F}$. By the definition of Laguerre homologies, the group G fixes each parallel class. Moreover, G is 2-transitive and effective on $\mathcal{F}$ and on each parallel class. Furthermore, G is 2-dimensional and, in fact, isomorphic to the affine group $x \mapsto ax + b$ for $a, b \in \mathbb{R}$, $a \neq 0$.

Clearly, every automorphism of $\mathcal{L}$ leaves $\mathcal{F}$ invariant. In particular, each automorphism in the connected component Γ^1 of the automorphism group Γ of $\mathcal{L}$ that contains the identity fixes $\mathcal{F}$. Furthermore, Γ^1 is transitive on $\mathcal{F}$ and on each parallel class. Let C_0, C_1 be two circles in $\mathcal{F}$ and let $\Sigma = (\Gamma_{C_0,C_1})^1$ be the connected component of the stabilizer Γ_{C_0,C_1} of the two circles. (Note that by connectedness, Σ is a subgroup of Γ^1.) Since C_0 and C_1 have 1-dimensional orbits under Γ (namely $\mathcal{F}$), we see that

$$\dim \Gamma = 2 + \dim \Sigma$$

by the dimension formula for Lie transformation groups (relating the dimensions of the group, orbits and stabilizers). Moreover, Σ acts effectively on C_0.

Let p_0 be a point on C_0 and consider the stabilizer $\Sigma_0 = \Sigma_{p_0}$. Then Σ_0 induces a group $\tilde{\Sigma}_0$ of collineations of the derived projective plane $\mathcal{P}_{p_0}$ of $\mathcal{L}$ at p_0. This group fixes the line W at infinity, the oval $\tilde{C}_1$ induced by the circle C_1 and the point w_0 at infinity of the line that comes from C_0. Since $\tilde{C}_1$ is a topological oval in $\mathcal{P}_{p_0}$, there are precisely two tangents to $\tilde{C}_1$ through w_0, one being W; compare [4, ?], [1, Satz 3.7.a] or [12, proof of 55.17]. Let L be the other tangent and let $q_1 = L \cap \tilde{C}_1$. Then $\tilde{\Sigma}_0$ fixes q_1 and so does Σ_0, because q_1 is a point of $\mathcal{L}$. Therefore Σ_0 fixes the four points p_0, $p_1 = C_1 \cap |p_0|$, q_1 and $q_0 = C_0 \cap |q_1|$. Note that the circle D_0 which induces L in $\mathcal{P}_p$ touches C_0 at p_0 and C_1 at q_1.

Let D_1 be the circle through p_1 that touches C_0 at q_0. If, for example, C_1 is above C_0 in Z, we see that $p_1 \in D_1$ is above $p_0 \in D_0$ in $|p_0|$ and $q_0 \in D_1$ is below $q_1 \in D_0$ in $|q_1|$. Hence, D_0 and D_1 must intersect in two points r_1 and r_2, and Σ_0 fixes $\{r_1, r_2\}$. However, r_1 and r_2 lie in different connected components of $Z \setminus \{|p_0|, |q_1|\}$ so that Σ_0 must fix each of r_1 and r_2 by connectedness of Σ.

The fact that the stabilizer of four points in a flat Laguerre plane three of which are on a circle and the fourth one is off that circle is trivial (compare [14]) shows that $\Sigma_0 = \{\mathrm{id}\}$. In particular, $\dim \Sigma_0 = 0$ so that, by the dimension formula, $\dim \Sigma \leq 1$ and thus Γ is at most 3-dimensional.

We now assume that Γ is 3-dimensional. From the arguments above we infer that Σ must be 1-dimensional. Furthermore, Σ is connected, transitive and effective on C_0. Given a circle C not in $\mathcal{F}$ there is a circle B in $\mathcal{F}$ that touches C from below. Using the group Γ, the circle B can be taken to the circle C_0, and $C \cap B$, the point of touching, to any particular point p_0 on C_0. Since the group of C_0-homologies is transitive on the set of circles $\neq C_0$ touching C_0 at p_0, we see that B can be taken to any particular circle $\neq C_0$ that touches C_0 at p_0 (for example, the circle D_0 from above). This shows that the circles in $\mathcal{C} \setminus \mathcal{F}$ form an orbit under Γ. □

We keep the notation in the proof of Theorem 4.1 and assume that Γ is 3-dimensional. We represent the cylinder Z as $\mathbb{S}^1 \times \mathbb{R}$ where $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. The coordinates obtained in this way differ from the ones usually used but are more convenient for us. We may choose any interval $[u, u+2\pi)$ of length 2π with its end points identified to represent $\mathbb{S}^1$. In particular, circles are represented by graphs of continuous periodic functions with period 2π.

As seen before, Σ is 1-dimensional, connected, transitive and effective on C_0. Hence, Σ is isomorphic to the rotation group $\mathrm{SO}_2(\mathbb{R})$ and, in fact, acts equivalently to the standard group of rotations on Z. We may therefore assume that the transformations in Σ are given by

$$(x, y) \mapsto (x + t, y)$$

for $t \in \mathbb{R}/2\pi\mathbb{Z}$. The circles in $\mathcal{F}$ are orbits under this group so that we obtain the circles

$$\{(x, a) \mid x \in \mathbb{S}^1\}$$

for $a \in \mathbb{R}$. As in the proof of Theorem 4.1 let G be the group generated by all Laguerre homologies at circles in $\mathcal{F}$. Since $\gamma \in G$ fixes each parallel class and $\sigma \in \Sigma$ fixes each circle in $\mathcal{F}$, we see that the commutator $\gamma^{-1}\sigma^{-1}\gamma\sigma$ fixes each parallel class and each circle in $\mathcal{F}$ and thus must be the identity. This shows that G and Σ commute. We may therefore assume that

$$(x, y) \mapsto (x + t, sy + a)$$

for $a, s \in \mathbb{R}$, $s \neq 0$, $t \in \mathbb{R}/2\pi\mathbb{Z}$, are automorphisms of $\mathcal{L}$.

Let $\{(x, f(x)) \mid x \in \mathbb{S}^1\}$ be a circle through $(\pi, 0)$ and $(0, 0)$ where $f : \mathbb{S}^1 \to \mathbb{R}$ is continuous but not identically 0. Then the circles of $\mathcal{L}$ are of the form

$$\{(x, sf(x + t) + a) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\}$$

for $a, s \in \mathbb{R}$, $t \in \mathbb{R}/2\pi\mathbb{Z}$. This shows that $\mathcal{L} = \mathcal{L}(f)$ is, in this case, completely determined by the single function f.

In particular, for $a = 0$, $s = 1$ and $t = \pi$ one has the circle

$$\{(x, f(x + \pi)) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\}$$

which passes through $(\pi, 0)$ and $(0, 0)\}$. Thus it must be of the form $y = sf(x)$ for a suitable $s \neq 0$, that is,

$$f(x + \pi) = sf(x) \ .$$

Applying this identity again for $f(x + \pi)$ one finds

$$f(x) = f(x + 2\pi) = sf(x + \pi) = s^2 f(x) \ .$$

Hence $s^2 = 1$ and thus $s = -1$ because a rotation through π takes the positive half of $y = f(x)$ to the negative half of $y = sf(x)$. Therefore

$$f(x + \pi) = -f(x) \ .$$

Since Γ is transitive on the points of $\mathcal{L}$, it follows that $\mathcal{L}$ is a Laguerre plane if and only if the derived incidence structure $\mathcal{A}$ of $\mathcal{L}$ at $(\pi, 0)$ is an affine plane. From the description of circles above one finds that the non-vertical lines of $\mathcal{A}$ are given by

$$y = s(f(x + u) + f(u))$$

for all $s \in \mathbb{R}$, $u \in \mathbb{R}/2\pi\mathbb{Z}$.

Note that the classical flat Laguerre plane admits a 3-dimensional group of automorphisms as described above. The usual parabola model of the classical flat Laguerre plane has point set $(\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ and circles $\{(x, ax^2 + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$ for $a, b, c \in \mathbb{R}$. According to [9, Proposition 2] there is a unique topology extending the natural topology of $\mathbb{R}^2$ such that one obtains a flat Laguerre plane. In this model such a group of transformations is, for example, given by

$$\left\{ (x, y) \mapsto \left(\frac{x \cos t - \sin t}{x \sin t + \cos t}, \frac{sy + a(x^2 + 1)}{(x \sin t + \cos t)^2} \right) \;\middle|\; a, s, t \in \mathbb{R}, s \neq 0 \right\} \ .$$

(In the first coordinate the usual conventions for linear fractional maps on $\mathbb{R} \cup \{\infty\}$ apply when dealing with the symbol ∞ or when dividing by 0.

Due to the way the infinite parallel class $\{\infty\} \times \mathbb{R}$ is topologically fitted into the cylinder in the parabola model, the behaviour in the second coordinate is less straightforward. For example, (∞, y) is taken to $(\cot t, \frac{sy+a}{\sin^2 t})$ in case $\sin t \neq 0$; an affine point (u, v) is close to (∞, y) if and only if u is close to ∞, that is, $|u|$ is large, and $\frac{v}{u^2+1}$ is close to y.) However, of course, the classical Laguerre plane has type VII. The coordinate transformation

$$(\mathbb{R} \cup \{\infty\}) \times \mathbb{R} \to (-\pi, \pi] \times \mathbb{R} : (x, y) \mapsto \begin{cases} (2\tan^{-1}(x), \frac{y}{x^2+1}), & \text{for } x \in \mathbb{R} \\ (\pi, y) & \text{for } x = \infty \end{cases}$$

takes a circle $\{(x, ax^2 + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$ to

$$\left\{ \left(u, \frac{b}{2} \sin u + \frac{c-a}{2} \cos u + \frac{c+a}{2} \right) \;\middle|\; u \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

and brings the group to the form we used above. Consequently, $f(x) = \sin x$ yields a Laguerre plane in the setting above, albeit the classical plane.

5. A model for a flat Laguerre plane of type V.A.1

In this section we are going to construct a flat Laguerre plane of Kleinewillinghöfer type V.A.1. We build on the information gained in the previous section, assuming that we have a 3-dimensional automorphism group. In order to obtain our model we modify the describing function of the classical real Laguerre plane. More precisely, we use

$$f(x) = \frac{\sin x}{1 + \sin^2 x} .$$

As in Section 4 our model for a Laguerre plane has point set $Z = \mathbb{S}^1 \times \mathbb{R}$ and circles

$$C_{a,t,b} = \{(x, af(x+t) + b) \mid x \in \mathbb{R}/2\pi\mathbb{Z}\}$$

for $a, b \in \mathbb{R}$, $t \in \mathbb{R}/\pi\mathbb{Z}$. We claim that the collection $\mathcal{C}$ of the above sets forms the circle set of a flat Laguerre plane $\mathcal{L}$.

Note that the parameters a and t are not uniquely determined by a circle. Indeed, $C_{-a,t,b} = C_{a,t+\pi,b}$, and $C_{0,0,b} = C_{0,t,b}$ for any t. The first of these coincidences is avoided by taking t modulo π (not 2π) as stated above. The second of the coincidences cannot be avoided. We often use $[-\frac{\pi}{2}, \frac{3\pi}{2})$ or $(-\pi, \pi]$ or any other convenient interval of length 2π with its endpoints identified to represent $\mathbb{S}^1$, the set of first coordinates of Z. Moreover, note that each circle is the graph of a continuous function from $\mathbb{S}^1 \approx \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$.

The circles $C_{0,0,b}$ for $b \in \mathbb{R}$ form a partition of the cylinder Z, that is,

$$\mathcal{F} = \{C_{0,0,b} \mid b \in \mathbb{R}\}$$

is a flock. Note that the restrictions on t made above ensure that each circle not in $\mathcal{F}$ uniquely determines its parameters a, t and b.

It is readily verified that the permutations

$$\gamma_{r,c,s} : (x, y) \mapsto (x + s, ry + c)$$

for $x \in \mathbb{R}/2\pi\mathbb{Z}$ and $r, c \in \mathbb{R}$, $r \neq 0$, $s \in \mathbb{R}/2\pi\mathbb{Z}$, are automorphisms of $\mathcal{L}$ and that $\gamma_{r,c,s}(C_{a,t,b}) = C_{ra,t-s,rb+c}$. Moreover, the permutation $\sigma : (x, y) \mapsto (-x, y)$ is an automorphism of $\mathcal{L}$ and $\sigma(C_{a,t,b}) = C_{-a,-t,b}$. The group Δ generated by these permutations of Z is transitive on Z and has two orbits on the set of circles, $\mathcal{F}$ and $\mathcal{C} \setminus \mathcal{F}$.

Before we come to the verification of the geometric axioms of a Laguerre plane we list some useful properties of the function f, that are straightforward to verify. f is periodic with period 2π and $f(x + \pi) = -f(x)$. Furthermore, f is infinitely often differentiable with

$$f'(x) = \frac{\cos^3 x}{(1 + \sin^2 x)^2}$$

and

$$f''(x) = -\frac{\sin x \cos^2 x(6 + \cos^2 x)}{(1 + \sin^2 x)^3} .$$

Proposition 5.1. *Two distinct circles intersect in at most two points.*

Proof. Let $C_1 \neq C_2$ be two circles. Since the automorphism group has two orbits on $\mathcal{C}$, it suffices to look at the following three cases.

$C_1, C_2 \in \mathcal{F}$: In this case the circles are disjoint because the circles in $\mathcal{F}$ form a partition of Z.

$C_1 \in \mathcal{F}$, $C_2 \in \mathcal{C} \setminus \mathcal{F}$: By using the group Δ we may assume that $C_1 = C_{0,0,b}$ and $C_2 = C_{1,0,0}$. We then have to solve the equation

$$f(x) = \frac{\sin x}{1 + \sin^2 x} = b .$$

For $b = 0$ one obviously obtains $C_1 \cap C_2 = \{(0, 0), (\pi, 0)\}$ and we have two points of intersection. For $b \neq 0$ we obtain the equation

$$\sin^2 x - \beta \sin x + 1 = 0$$

where we have written $\beta = \frac{1}{b}$.

The quadratic polynomial in $\sin x$ on the left-hand side has discriminant $\beta^2 - 4$. If $|\beta| < 2$, then we have no solutions. For $|\beta| = 2$, we have $\sin x = \frac{\beta}{2} = \pm 1$, and we obtain precisely one solution in $\mathbb{S}^1$. If $|\beta| > 2$, then $\sin x = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}$. However, $\frac{\beta + \sqrt{\beta^2 - 4}}{2} > 1$ for $\beta > 2$ and $\frac{\beta - \sqrt{\beta^2 - 4}}{2} < -1$ for

$\beta < -2$, which is not admissible. Therefore $\sin x = \frac{\beta - \mathrm{sign}(\beta)\sqrt{\beta^2-4}}{2}$, and this then yields precisely two solutions in $\mathbb{S}^1$. Hence, $|C_1 \cap C_2| \leq 2$.

$C_1, C_2 \in \mathcal{C} \backslash \mathcal{F}$: By using the group Δ and the symmetry between the two circles we may assume that $C_1 = C_{a,0,b}$ and $C_2 = C_{1,t,0}$. Note that $a \neq 0$ because $C_1 \notin \mathcal{F}$. Furthermore, we may assume that $0 < t < \pi$ because $t = 0$ leads to no point of intersection in case $a = 1$ and to at most two points of intersection in case $a \neq 1$. (For $a \neq 1$ one is led to the equation $f(x) = \frac{b}{1-a}$, which we know has at most two solutions as seen in the case $C_1 \in \mathcal{F}$, $C_2 \in \mathcal{C} \setminus \mathcal{F}$ above.) We then have to solve an equation of the form

$$f(x+t) - af(x) = b \ .$$

Let $g_{a,t}(x) = f(x+t) - af(x)$ denote the function on the left-hand side of the above equation. Then $g_{a,t}$ is a bounded, continuously differentiable function and thus has a maximum and a minimum. At the extremal points the derivative of $g_{a,t}$ must be zero. Since $f(x+\pi) = -f(x)$, one obtains that $g'_{a,t}(x+\pi) = -g'_{a,t}(x)$. Hence, the roots of the derivative come in pairs, one in $[-\frac{\pi}{2}, \frac{\pi}{2})$ and one in $[\frac{\pi}{2}, \frac{3\pi}{2})$. We want to show that $g'_{a,t}(x) = 0$ has exactly one solution x_0 in $[-\frac{\pi}{2}, \frac{\pi}{2})$. Once this is verified and, say $g'_{a,t}(-\frac{\pi}{2}) > 0$, then $g_{a,t}(x)$ is strictly increasing between $x_0 - \pi$ and x_0 and strictly decreasing from x_0 to $x_0 + \pi$. Thus $g_{a,t}(x) = b$ has at most two solutions. Hence, the two circles C_1 and C_2 have at most two points in common.

To verify our claim on the zeros of $g'_{a,t}(x)$ in $[-\frac{\pi}{2}, \frac{\pi}{2})$ note that $f'(\pm\frac{\pi}{2}) = 0$ and that $f'(x) > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. This implies that $g'_{a,t}(-\frac{\pi}{2}) \neq 0$. Then $g'_{a,t}(x) = 0$ if and only if $\frac{f'(x+t)}{f'(x)} = a$. It therefore suffices to show that the function $h_t(x) = \frac{f'(x+t)}{f'(x)}$ is a bijection from $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$. Since $f'(-\frac{\pi}{2} + t) > 0$ and $f'(\frac{\pi}{2} + t) < 0$ (note that $0 < t < \pi$), we see that $\lim_{x \to -\frac{\pi}{2}^+} h_t(x) = +\infty$ and $\lim_{x \to \frac{\pi}{2}^-} h_t(x) = -\infty$. Hence, $h_t(x)$ is surjective.

For the injectivity of $h_t(x)$ consider $h'_t(x) = \frac{f''(x+t)f'(x) - f'(x+t)f''(x)}{(f'(x))^2}$. Clearly, h'_t has zero at $\frac{\pi}{2} - t$ in I. In order to find other possible zeros of h'_t in I we follow an idea of one of the referees for a shorter proof than the one we originally had. We consider

$$F(x) = -\frac{f''(x)}{f'(x)} = \frac{(6 + \cos^2 x)\sin x}{(1 + \sin^2 x)\cos x} = G(\tan x)$$

where G is the rational function

$$G(u) = \frac{(7 + 6u^2)u}{1 + 2u^2} = 3u + \frac{4u}{1 + 2u^2} \ .$$

Now

$$G'(u) = \frac{7 + 4u^2 + 12u^4}{(1 + 2u^2)^2} > 0$$

and thus $F'(x) = G'(\tan x)(1 + \tan^2 x) > 0$. This shows that F is injective.

We now assume that $x_0 \neq \frac{\pi}{2} - t$ is a zero of h'_t in I. Then $f'(x_0), f'(x_0 + t) \neq 0$, and we have that $F(x_0 + t) = F(x_0)$. However, as seen before, F is injective in I so that the above equation is impossible in I. On the other hand, if $x_0 + t > \frac{\pi}{2}$, then $x_0 + t - \pi \in I$. But F is periodic with period π so that $F(x_0 + t - \pi) = F(x_0 + t) = F(x_0)$, which is again a contradiction to the injectivity of F.

This shows that $\frac{\pi}{2} - t$ is the only zero of h'_t in the interval I. Hence, one finds that $h'_t(x) < 0$ for all $x \in I \setminus \{\frac{\pi}{2} - t\}$, and we have verified that $h_t(x)$ is injective and thus a bijection. □

In order to verify that $\mathcal{L}$ is a Laguerre plane it suffices, as a consequence of the transitivity of the group Δ on the point set, to show that the derived incidence structure $\mathcal{A} = \mathcal{A}_{(\pi,0)}$ at the point $(\pi, 0)$ is an affine plane. The non-vertical lines of $\mathcal{A}$ are the circles $C_{a,t,af(t)}$ minus the point $(\pi, 0)$. Thus the non-vertical lines $L_{a,t}$ of $\mathcal{A}$ are given by

$$y = a(f(x + t) + f(t)) \; .$$

Proposition 5.2. *Two distinct points in $\mathcal{A}$ are joined by a unique line.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two given points of $\mathcal{A}$. If $x_1 = x_2$, then the vertical line $x = x_1$ is the only line of $\mathcal{A}$ joining the two points.

We now assume that $x_1 \neq x_2$. We then have to solve the system of equations

$$\begin{aligned} y_1 &= a(f(x_1 + t) + f(t)) \qquad \text{and} \\ y_2 &= a(f(x_2 + t) + f(t)) \end{aligned}$$

for a and t.

If $y_1 = y_2 = 0$, then $a = 0$ gives us a solution. In all other cases we know that $a \neq 0$ so that t is a solution of

$$y_2(f(x_1 + t) + f(t)) - y_1(f(x_2 + t) + f(t)) = 0 \; .$$

Let $h(t)$ denote the function of t on the left-hand side of the above equation. Then

$$\begin{aligned} h(0) &= y_2 f(x_1) - y_1 f(x_2) \qquad \text{and} \\ h(\pi) &= y_2 f(x_1 + \pi) - y_1 f(x_2 + \pi) = y_1 f(x_2) - y_2 f(x_1) = -h(0) \; . \end{aligned}$$

Hence, for $h(0) \neq 0$, there is at least one solution t in the interval $(0, \pi)$ of the above equation by the continuity of h. If $h(0) = 0$, then of course $t = 0$ is a solution.

Since not both of y_1 and y_2 are 0, at least one of $f(x_1 + t) + f(t)$ or $f(x_2 + t) + f(t)$ is non-zero, and we obtain $a = \frac{y_i}{f(x_i+t)+f(t)}$ for such an $i = 1, 2$. The uniqueness of the resulting line then follows from Proposition 5.1. □

PROPOSITION 5.3. *Two non-vertical lines L_{a_1,t_1} and L_{a_2,t_2} in $\mathcal{A}$ are parallel if and only if*

$$a_1 f'(t_1) = a_2 f'(t_2) \ .$$

Moreover, the parallel axiom is satisfied in $\mathcal{A}$.

PROOF. We first assume that L_{a_1,t_1} and L_{a_2,t_2} are two non-vertical lines in $\mathcal{A}$ such that $a_1 f'(t_1) \neq a_2 f'(t_2)$. These lines come from the circles $C_1 = C_{a_1,t_1,a_1 f(t_1)}$ and $C_2 = C_{a_2,t_2,a_2 f(t_2)}$, respectively. By assumption functions describing these circles have different derivatives at $x = \pi$ so that the two curves locally intersect transversally, that is, there are points on C_1 near $(\pi, 0)$ that are in different connected components of $Z \setminus C_2$. But $C_1 \setminus \{(\pi, 0)\} \approx \mathbb{R}$ is connected so that C_1 and C_2 must intersect in at least another point $\neq (\pi, 0)$. For the lines in $\mathcal{A}$ this means that they must also intersect in $\mathcal{A}$ and therefore are not parallel.

We now consider an $m \in \mathbb{R}$ and a point $(x, y) \in P = (-\pi, \pi) \times \mathbb{R}$. We claim that there is a unique line $L_{a,t}$ that passes through (x, y) and such that $m = -af'(t)$. Once this claim is verified it follows that the lines $L_{a,t}$ with the same values of $af'(t)$ form a partition of P. In particular, two such lines are parallel, and the parallel axiom is satisfied in $\mathcal{A}$.

We begin with the case $m = 0$. Then $a = 0$ or $t = \frac{\pi}{2}$. In the former case we have the line $L_{0,0}$ and in the latter case we have

$$a = \frac{y}{f(x + \frac{\pi}{2}) + f(\frac{\pi}{2})} \ .$$

Note that

$$f\left(x + \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \frac{\cos x}{1 + \cos^2 x} + \frac{1}{2} = \frac{(1 + \cos x)^2}{2(1 + \cos^2 x)} > 0$$

for all $-\pi < x < \pi$ so that a is well defined. Furthermore, the fact that $f(x + \frac{\pi}{2}) + f(\frac{\pi}{2}) > 0$ for all $x \in (-\pi, \pi)$ implies that the lines $L_{a,\frac{\pi}{2}}$ for $a \in \mathbb{R}$ form a partition of P.

We now assume that $m \neq 0$ and thus $t \neq \frac{\pi}{2}$. Then $a = -\frac{m}{f'(t)}$ and we have to find $-\frac{\pi}{2} < t < \frac{\pi}{2}$, such that

$$y_m = -\frac{y}{m} = \frac{f(x + t) + f(t)}{f'(t)} \ .$$

Let $h_x(t)$ be the function on the right-hand side. Then, because as seen above $f(x + \frac{\pi}{2}) + f(\frac{\pi}{2}) > 0$, we have that $\lim_{t \to +\frac{\pi}{2}^-} h_x(t) = +\infty$ and $\lim_{t \to -\frac{\pi}{2}^+} h_x(t) = -\infty$. Hence, by the continuity of h_x on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain that for each y_m there is at least one $-\frac{\pi}{2} < t < \frac{\pi}{2}$ such that $y_m = h_x(t)$. What remains to be shown is uniqueness. We do this by verifying that h_x is injective. In fact, we want to show that h_x is strictly increasing.

Explicitly we have

$$\begin{aligned} h_x(t) &= \frac{f(x+t)+f(t)}{f'(t)} \\ &= \frac{(\sin(x+t)+\sin t)(1+\sin t\sin(x+t))(1+\sin^2 t)}{(1+\sin^2(x+t))\cos^3 t} \ . \end{aligned}$$

We make the substitutions

$$u = \tan t \qquad \text{and} \qquad v = \tan\left(\frac{x}{2}\right)$$

(so that $\sin x = \frac{2v}{1+v^2}$ and $\cos x = \frac{1-v^2}{1+v^2}$) and use the addition formula for sine to obtain

$$H_v(u) = 2\frac{(u+v)((u+v)^2+1+u^2)(1+2u^2)}{(1+v^2)^2(1+u^2)+(2v+(1-v^2)u)^2} \ .$$

Then H_v has derivative

$$H_v'(u) = \frac{2p(u,v)}{((1+v^2)^2(1+u^2)+(2v+(1-v^2)u)^2)^2}$$

where

$$\begin{aligned} p(u,v) = {} & 24(v^4+1)u^6 + 32(v^5-2v^3+3v)u^5 \\ & +(12v^6-68v^4+228v^2+28)u^4 + (-16v^5+208v^3+64v)u^3 \\ & +(4v^6+102v^4+108v^2+10)u^2 + (32v^5+72v^3+8v)u \\ & +7v^6+19v^4+5v^2+1 \end{aligned}$$

If we can show that $p(u,v) \geq 1$ for all $u,v \in \mathbb{R}$, then $H_v'(u) > 0$ and thus $h_x'(t) > 0$, and our claim on the monotonicity of h_x is verified.

Note that $p(u,0) = 24u^6 + 28u^4 + 10u^2 + 1 \geq 1$ and $p(0,v) = 7v^6 + 19v^4 + 5v^2 + 1 \geq 1$ and that these polynomials have value 1 if and only $u = 0$ and $v = 0$, respectively. Furthermore, there are no linear terms in $p(u,v)$ so that the graph of p has a horizontal tangent plane at (0,0,1). The quadratic terms of $p(u,v)$ are $10u^2+8uv+5v^2$. Since this quadratic form is non-degenerate and positive definite, $p(u,v)$ has a local minimum at (0,0).

We consider $r(u,v) = p(u,v) - (1+10u^2+8uv+5v^2)$. For $u,v \neq 0$, let $z = \frac{v}{u}$. Then

$$\begin{aligned} \frac{r(u,zu)}{u^4} = {} & (24z^4+32z^5+12z^6)u^6 + (-16z^5-68z^4-64z^3+4z^6)u^4 \\ & +(24+7z^6+102z^4+228z^2+208z^3+96z+32z^5)u^2 \\ & +28+19z^4+108z^2+64z+72z^3 \ . \end{aligned}$$

By substituting w for u^2 in the above we obtain the cubic polynomial

$$\begin{aligned} r_z(w) &= 4z^4(3z^2+8z+6)w^3 + 4z^3(z^3-4z^2-17z-16)w^2 \\ &\quad +(7z^6+32z^5+102z^4+208z^3+228z^2+96z+24)w \\ &\quad +19z^4+72z^3+108z^2+64z+28 \end{aligned}$$

(so that $r(u, zu) = u^4 r_z(u^2)$). Then

$$\begin{aligned} r_z'(w) &= 12z^4(3z^2+8z+6)w^2 + 8z^3(z^3-4z^2-17z-16)w \\ &\quad +7z^6+32z^5+102z^4+208z^3+228z^2+96z+24 \end{aligned}$$

and this quadratic polynomial in w has discriminant $-16z^4D(z)$ where

$$\begin{aligned} D(z) &= 59z^8+488z^7+1884z^6+4480z^5+7212z^4 \\ &\quad +7904z^3+5600z^2+2304z+432\ . \end{aligned}$$

We make the substitution $z = z_1 - 1$ in order to reduce the magnitude of the coefficients in $D(z)$. Then

$$\begin{aligned} d(z_1) &= D(z_1-1) \\ &= 59z_1^8+16z_1^7+120z_1^6+120z_1^5+122z_1^4-48z_1^3+24z_1^2+8z_1+11\ . \end{aligned}$$

From

$$d(z_1) = (z_1+1)^2(8z_1^6+60z_1^4+4)+12z_1^2(2z_1-1)^2+51z_1^8+52z_1^6+14z_1^4+8z_1^2+7$$

we see that $d(z_1) \geq 7$ and thus $D(z) \geq 7$. Therefore, the discriminant of $r_z'(w)$ is also always negative for $z \neq 0$. Since the coefficient of w^2 in $r_z'(w)$ is $12z^4(3z^2+8z+6) > 0$ for $z \neq 0$, we see that $r_z'(w) \geq 0$ for all w and all $z \neq 0$. Hence $r_z(w)$ is strictly increasing in w. But

$$\begin{aligned} r_z(0) &= 19z^4+72z^3+108z^2+64z+28 \\ &= \frac{1}{19}(19z+36)^2z^2 + \frac{428}{133}z^2 + \frac{4}{7}(8z+7)^2 > 0 \end{aligned}$$

so that $r_z(w) > 0$ for all $w \geq 0$. Therefore $r(u, zu) \geq 0$ for all u and z. This then implies $r(u, v) \geq 0$ and thus $p(u, v) \geq 1 + 10u^2 + 8uv + 5v^2 \geq 1$ for all $u, v \neq 0$. Together with the previous result on $p(u, 0)$ and $p(0, v)$ we thus obtain that $p(u, v) \geq 1$ for all u and v. □

PROPOSITION 5.4. *The derived incidence structure $\mathcal{A} = \mathcal{A}_{(\pi,0)}$ of $\mathcal{L}$ at $(\pi, 0)$ is a non-Desarguesian affine plane.*

PROOF. From Propositions 5.1 and 5.2 we know that $\mathcal{A}$ is a linear space. Proposition 5.3 shows that the parallel axiom holds in $\mathcal{A}$. Hence, $\mathcal{A}$ is an affine plane.

It remains to show that $\mathcal{A}$ is non-Desarguesian. To see this consider the two triangles with vertices

$$\begin{array}{lllll} p_1 & = & (0,-20\sqrt{3}), & p_2 = \left(\frac{2\pi}{3},0\right), & p_3 = \left(-\frac{\pi}{3},0\right) \quad \text{and} \\ p_1' & = & (0,0), & p_2' = \left(\frac{2\pi}{3},\frac{20}{49}\sqrt{3}\right), & p_3' = \left(-\frac{\pi}{3},\frac{36}{7}\sqrt{3}\right), \end{array}$$

respectively. The sides in these triangles are

$$\begin{aligned} p_1p_2 &= L_{35,-\frac{\pi}{3}}, \\ p_1p_3 &= L_{-25\sqrt{3},\frac{\pi}{6}}, \\ p_1'p_2' &= L_{\frac{10}{7},0}, \\ p_1'p_3' &= L_{-18,0}. \end{aligned}$$

Furthermore, the lines p_ip_i' are parallel. (These are the parallel classes $|p_1| = \{0\} \times \mathbb{R}$, $|p_2| = \{\frac{2\pi}{3}\} \times \mathbb{R}$ and $|p_3| = \{-\frac{\pi}{3}\} \times \mathbb{R}$.)

It is easily checked with Proposition 5.3 that the corresponding sides p_1p_j and $p_1'p_j'$ of the triangles are parallel for $j = 2, 3$. However, the third pair of sides $p_2p_3 = L_{0,0}$ and $p_2'p_3'$ is not parallel by again Proposition 5.3. (Lines parallel to $L_{0,0}$ are of the form $L_{a,\frac{\pi}{2}}$, which intersect $|p_2|$ and $|p_3|$ in $(\frac{2\pi}{3}, \frac{a}{10})$ and $(-\frac{\pi}{3}, \frac{9a}{10})$, respectively. But the second coordinate of p_3' is not 9 times the second coordinate of p_2'.)

This shows that $\mathcal{A}$ is not Desarguesian. □

Theorem 5.5. *The incidence structure $\mathcal{L}$ is a flat Laguerre plane of Kleinewillinghöfer type V.A.1.*

Proof. We know from the transitivity properties of $\mathcal{L}$ and from Proposition 5.4 that $\mathcal{L}$ is a Laguerre plane and thus a flat Laguerre plane by the continuity of f.

It is readily verified that $\{\gamma_{s,(1-s)b,0} \mid s \in \mathbb{R}, s \neq 0\}$ is a linearly transitive group of $C_{0,0,b}$-homologies. Hence, the set $\mathcal{Z}$ of all circles C for which the automorphism group of $\mathcal{L}$ is linearly transitive (with respect to C-homologies) contains a flock of circles as in type V.

By [11, Proposition 3.4] the only set $\mathcal{Z}$ containing a flock of circles plus an extra circle is as in type VII, and such a flat Laguerre plane is ovoidal by [11, Corollary 3.2]. However, the derived plane $\mathcal{A}_{(\infty,0)}$ at $(\infty, 0)$ is non-Desarguesian by Proposition 5.4 and we have a contradiction. Hence, $\mathcal{L}$ must be of Kleinewillinghöfer type V. It follows from the list of combined types, see [11, Theorem 6.1], that the combined type then is V.A.1. □

Theorem 5.6. *Each automorphism of $\mathcal{L}$ is of the form $\gamma_{r,c,s}$ or $\gamma_{r,c,s}\sigma$. Hence, the group Δ is the full automorphism group of $\mathcal{L}$.*

PROOF. Let φ be an automorphism of $\mathcal{L}$. But $\mathcal{L}$ is a flat Laguerre plane so that φ is continuous and even a homeomorphism of Z. Since $\mathcal{L}$ has type V, the flock $\mathcal{F}$ must be must invariant. We may therefore assume that, up to automorphisms of $\mathcal{L}$ of the form $\gamma_{r,c,0}$, the circles $C_{0,0,0}$ and $C_{0,0,1}$ in $\mathcal{F}$ are fixed by φ. Using an automorphism of the form $\gamma_{1,0,s}$ we may further assume that the point $(\pi, 0)$ on $C_{0,0,0}$ is fixed by φ. But then $(\pi, 1)$ is fixed as well, because φ permutes the parallel classes of $\mathcal{L}$.

In order to obtain more fixed points, we now employ an argument similar to the one used in the proof or Theorem 4.1. The circle $C_{0,0,0}$ induces a line in the derived projective plane $\mathcal{P}$ at $(\pi, 0)$, and the other circle $C_{0,0,1}$ induces a closed oval $\mathcal{O}$ in $\mathcal{P}$. Since $\mathcal{P}$ is a (topological, compact) 2-dimensional projective plane, there is exactly one line in $\mathcal{P}$ other than the line at infinity that is a tangent to $\mathcal{O}$ and passes through the point at infinity of the line induced by $C_{0,0,0}$; compare [4, 2.5], [1, Satz 3.7.a] or [12, proof of 55.17]. In the coordinates of our Laguerre plane this is the circle $C_{1,\pi/2,1/2}$; this circle touches $C_{0,0,0}$ at $(\pi, 0)$ and $C_{0,0,1}$ at $(0, 1)$. Hence the points $(0, 0)$ and $(0, 1)$ are fixed by φ. If necessary, we may use the automorphism σ to achieve that each connected component of $Z \setminus (\{0, \pi\} \times \mathbb{R})$ (the complement of the parallel classes of $(0, 0)$ and $(\pi, 0)$) is left invariant.

Now, the automorphism $\gamma_{-1,0,1}$ interchanges the circles $C_{0,0,0}$ and $C_{0,0,1}$ and therefore $C_{-1,\pi/2,1/2} = \gamma_{-1,0,1}(C_{1,\pi/2,1/2})$ touches $C_{0,0,0}$ at $(\pi, 1)$ and $C_{0,0,1}$ at $(0, 0)$. Hence, φ stabilizes both circles $C_{1,\pi/2,1/2}$ and $C_{1,\pi/2,1/2}$ and thus also their intersection $C_{1,\pi/2,1/2} \cap C_{1,\pi/2,1/2} = \{(\frac{\pi}{2}, \frac{1}{2}), (\frac{3\pi}{2}, \frac{1}{2})\}$. Since these two points are in different connected components of $Z \setminus (\{0, \pi\} \times \mathbb{R})$, each of which is invariant under φ by assumption, we see that each point $(\frac{\pi}{2}, \frac{1}{2})$ and $(\frac{3\pi}{2}, \frac{1}{2})$ must be fixed by φ.

However, by [10, Lemma 5.4.2], the identity is the only automorphism of a flat Laguerre plane that fixes three points on a circle and a fourth point off this circle. Thus, by using elements in Δ, we have reduced φ to the identity. Hence, every automorphism of $\mathcal{L}$ is an element of Δ and thus is of the form $\gamma_{r,c,s}$ or $\gamma_{r,c,s}\sigma$. □

Note that $\{\gamma_{r,c,s} \mid r, s \in \mathbb{R}, r > 0, c \in \mathbb{R}/2\pi\mathbb{Z}\}$ is a 3-dimensional connected subgroup of $\Delta = \Gamma$, and thus must be the connected component Γ^1 of Γ that contains the identity. From this one sees that Γ^1 has index 4 in Γ.

An obvious generalization of the Laguerre plane $\mathcal{L}$ is obtained in the following way. For $0 \leq r \leq 1$ define

$$f_r(x) = \frac{\sin x}{1 + r \sin^2 x}$$

and use this function generate an incidence structure as in section 4 using the 3-dimensional group Δ. Note that for $r = 0$ we obtain the classical real Laguerre plane, and above we have seen that for $r = 1$ we obtain a non-classical flat Laguerre plane of Kleinewillinghöfer type V.A.1.

A lot of additional experimentation and numerical evidence in Maple along with the motivation given below suggest the following.

CONJECTURE. For $0 < r \leq 1$ the incidence structures $\mathcal{L}(f_r)$ are mutually non-isomorphic flat Laguerre planes of Kleinewillinghöfer type V.A.1.

We sometimes worked in a different setting, using a coordinate transformation. The cylinder is then represented as $(\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$ (as in the parabola model of the classical real Laguerre plane), Consequently, circles and automorphisms are described by formulae that involve only rational functions. Apart from facilitating some of the computations, this second perspective might help to prove the conjecture.

Let $x = 2\tan^{-1}(u)$. Then, up to the scalar factor 2, the function f_r becomes

$$g_q(u) = \frac{u(u^2+1)}{u^4 + qu^2 + 1}$$

where $2 \leq q \leq 6$. The admissible values $q = 2$ and $q = 6$ correspond to $r = 0$ and $r = 1$, respectively; in fact, $r = \frac{q-2}{4}$. With the function g_q circles are the graphs of $u \mapsto ag_q\left(\frac{u-t}{tu+1}\right) + b$ for $a, b, t \in \mathbb{R}$.

In either setting many of the steps in the verification of the axioms of a Laguerre plane or the verification of type V.A.1 go through as for $\mathcal{L}$ except for showing the injectivity of $h_t(x)$ in the last case in the proof of Proposition 5.1 and the injectivity of $h_x(t)$ in the proof of Proposition 5.3. In these latter cases we are led to certain polynomial equations, and we have to check that the corresponding polynomials do not have more than two zeros. In the cases $q = 2$ and $q = 6$ these polynomials have a special form and an analytic solution can be found. However, we failed to prove the general case $2 < q < 6$ analytically, the main reason being the impossibility to solve general polynomial equations of degree five and higher, which arise in our setting with coefficients depending on several parameters.

We have verified the conjecture for several $q \in \mathbb{Q}$, $2 \leq q \leq 6$ using a procedure in Maple, called Realrootcount. The underlying algorithm relies on Theorem 2.1 of [18] and guarantees to give the correct number of real solutions of a system of polynomial equations with rational coefficients. Of course, this method can never exhaustively check all rational numbers in our range. However, if one could show the conjecture for all admissible rational numbers, then using the density of $\mathbb{Q}$ in $\mathbb{R}$ along with continuity would prove the conjecture.

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