A characterization of quadrics by intersection numbers

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Abstract

This work is inspired by a paper of Hertel and Pott on maximum non-linear functions [8]. Geometrically, these functions correspond with quasi-quadrics; objects introduced in [5]. Hertel and Pott obtain a characterization of some binary quasi-quadrics in affine spaces by their intersection numbers with hyperplanes and spaces of codimension 2.
We obtain a similar characterization for quadrics in projective spaces by intersection numbers with low-dimensional spaces. Ferri and Tallini [7] characterized the non-singular quadric $Q(4,q)$ by its intersection numbers with planes and solids. We prove a corollary of this theorem for $Q(4,q)$ and then extend this corollary to all quadrics in $PG(n,q), n \geq 4$. The only exceptions we get occur for q even, where we can have an oval or an ovoid as intersection with our point set in the non-singular part.

1 Notations and background

1.1 Polar spaces and generalized quadrangles

Polar spaces were first described axiomatically by Veldkamp [12]. Later on, Tits simplified Veldkamp's list of axioms and further completed the theory [11]. We recall Tits' definition of polar spaces.
A polar space of rank n, $n \geq 2$, is a point set P together with a family of subsets of P called subspaces, satisfying the following axioms.

(i) A subspace, together with the subspaces it contains, is a d-dimensional projective space with $-1 \leq d \leq n-1$ (d is called the dimension of the subspace).

(ii) The intersection of two subspaces is a subspace.

(iii) Given a subspace V of dimension $n-1$ and a point $p \in P \backslash V$, there is a unique subspace W of dimension $n-1$ such that $p \in W$ and $V \cap W$ has dimension $n-2$; W contains all points of V that are joined to p by a line (a line is a subspace of dimension 1).

(iv) There exist two disjoint subspaces of dimension $n-1$.

The finite classical polar spaces are the following structures.

(i) The non-singular quadrics in odd dimension, $Q^+(2n+1,q)$ and $Q^-(2n+1,q)$, together with the subspaces they contain, giving a polar space of rank $n+1$ and n respectively. The non-singular parabolic quadrics in even dimension, $Q(2n,q)$, together with the subspaces they contain, giving a polar space of rank n.

(ii) The non-singular hermitian varieties in $PG(2n,q^2)$, together with the subspaces they contain, $n \geq 2$ (respectively, $PG(2n+1,q^2)$, $n \geq 1$), giving a polar space of rank n (respectively, rank $n+1$).

(iii) The points of $PG(2n+1,q)$, together with the totally isotropic subspaces of a non-singular symplectic polarity of $PG(2n+1,q)$, giving a polar space of rank n.

By theorems of Veldkamp and Tits, all polar spaces with finite rank at least 3 are classified. In the finite case (i.e. the polar space has a finite number of points), we get the following theorem, which can be found in [11].

Theorem 1.1 *A finite polar space of rank at least 3 is classical.*

Buekenhout and Shult described polar spaces as point-line geometries, and it is this description we will use.

Definition A Shult space is a point-line geometry $S=(P,B,I)$, with B a non-empty set of subsets of P of cardinality at least 2, such that the incidence relation I (which is containment here) satisfies the following axiom. For each line $L \in B$ and for each point $p \in P \backslash L$, the point p is collinear with either one or all points of the line L.

A Shult space is non-degenerate if no point is collinear with all other points. A Shult space is linear if two distinct lines have at most one common point. Buekenhout and Shult proved the following fundamental theorem [4].

Theorem 1.2 *(i) Every non-degenerate Shult space is linear.*

(ii) If S is a non-degenerate Shult space of finite rank at least 3, and if all lines contain at least three points, then the Shult space together with all its subspaces is a polar space.

A finite generalized quadrangle (GQ) of order (s,t) is an incidence structure $S = (P,B,I)$ in which P and B are disjoint (non-empty) sets of objects called points and lines respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(GQ3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q,M) such that $pIMIqIL$.

A generalized quadrangle (GQ) of order (s,t) contains $(s+1)(st+1)$ points. If $s=t$, then S is also said to be of order s.
If S has a finite number of points and if $s>1$ and $t>1$, then it is easy to show that one can replace axiom (GQ1) by the following axioms.

(GQ1') No point is collinear with all points.

(GQ1") There is a point on at least two lines.

It is this alternative definition which we will use in our proofs.

1.2 The classical generalized quadrangles

Consider a non-singular quadric of Witt index 2, that is, of projective index 1, in $PG(3,q)$, $PG(4,q)$ and $PG(5,q)$ respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $Q^+(3,q)$, $Q(4,q)$ and $Q^-(5,q)$ respectively, and of order $(q,1)$, (q,q) and (q,q^2) respectively. Next, let H be a non-singular hermitian variety in $PG(3,q^2)$, respectively $PG(4,q^2)$. The points and lines of H form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $PG(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ, denoted $W(q)$, of order q. The generalized quadrangles defined here are the so-called classical generalized quadrangles.

Definition A generalized quadrangle $S=(P,B,I)$ is fully embedded in a projective space $PG(V)$ if there is a map π from P (respectively B) to the set of points (respectively lines) of a projective space $PG(V)$, V a vector space over some skew field (not necessarily finite-dimensional), such that:

(i) π is injective on points,

(ii) if $x \in P$ and $L \in B$ with xIL, then $x^\pi I L^\pi$,

(iii) the set of points x^π, where $x \in P$, generates $PG(V)$,

(iv) every point in $PG(V)$ on the image of a line of the quadrangle is also the image of a point of the quadrangle.

The following beautiful theorem is due to Buekenhout and Lefèvre [3].

Theorem 1.3 *Every finite generalized quadrangle fully embedded in projective space is classical.*

A lot of information on finite generalized quadrangles can be found in the reference work [9].

1.3 Other background

Definition A blocking set with respect to t-spaces in $PG(n,q)$ is a set B of points such that every t-dimensional subspace of $PG(n,q)$ meets B in at least one point.

The following result of Bose and Burton gives a nice characterization of the smallest ones [2].

Theorem 1.4 *If B is a blocking set with respect to t-spaces in $PG(n,q)$, then $|B| \geq |PG(n-t,q)|$ and equality holds if and only if B is an $(n-t)$-dimensional subspace.*

Definition A k-arc of $PG(2,q)$ is a set of k points, no three collinear. Let $m(2,q)$ denote the maximal size of a k-arc in $PG(2,q)$.

We state the Bose result on the maximum size of a k-arc in $PG(2,q)$ [1].

Theorem 1.5 *If q is odd, then $m(2,q) = q+1$. If q is even, then $m(2,q) = q+2$.*

Definition A k-cap in $PG(n,q)$ is a set of k points in $PG(n,q)$, no three of which are collinear.

The size of a k-cap in $PG(3,q)$ is bounded. For q even in [1] and for q odd in [10].

Theorem 1.6 *If K is a k-cap of $PG(3,q)$, then $k \leq q^2+1$ for $q > 2$, and $k \leq 8$ for $q = 2$.*

Definition A (q^2+1)-cap of $PG(3,q)$, $q > 2$, is called an ovoid; an ovoid of $PG(3,2)$ is a set of 5 points of $PG(3,2)$ no four of which are coplanar. A $(q+1)$-arc of $PG(2,q)$ is called an oval.

Lemma 1.7 *Consider a set K of points in $PG(4,q)$. Suppose all planes intersect K in 1, $q+1$ or $2q+1$ points. If K is a cap in $PG(4,q)$, then $|K| \leq q^3+1$.*

Proof Consider a line L intersecting K in 2 points and consider all planes through L in $PG(4,q)$. These planes can not intersect K in $2q+1$ points, by theorem 1.5. Hence, K contains at most

$$(q^2+q+1)(q-1)+2 = q^3+1$$

points. ∎

1.4 Previous characterization results

We state the following result of Durante, Napolitano and Olanda [6].

Theorem 1.8 *Let K be a set of points in $PG(3,q)$, with $|K| = q^2+q+1$, and suppose that K contains at least two lines. Furthermore suppose that K intersects every plane in 1, $q+1$ or $2q+1$ points. Then K is a cone projecting an oval in a plane Π from a point v not in Π.*

Ferri and Tallini proved the following nice characterization of the parabolic quadric $Q(4,q)$ [7].

Theorem 1.9 *A set K of points in $PG(n,q)$, with $n \geq 4$ and $|K| \geq q^3+q^2+q+1$, intersecting all planes in 1, a or b points, where $b \geq 2q+1$, and intersecting every solid in c, $c+q$ or $c+2q$ points, where $c \leq q^2+1$, such that solids intersecting in c and solids intersecting in $c+q$ points exist, is a non-singular quadric of $PG(4,q)$.*

2 A corollary of the theorem of Ferri and Tallini

We consider a set K of points in $PG(4,q)$ intersecting every plane in 1, $q+1$ or $2q+1$ points, and every solid in q^2+1, q^2+q+1 or q^2+2q+1 points.
We will call planes intersecting K in 1, $q+1$ and $2q+1$ points respectively, small, medium and large respectively. We will call solids intersecting K in q^2+1, q^2+q+1 and q^2+2q+1 points respectively, small, medium and large respectively.
We prove the conditions required for the characterization by Ferri and Tallini of $Q(4,q)$. Consider a given solid Π. We will count how many small, medium and large planes respectively there are in Π; call the number of them a, b and c respectively. Denote the number of points of K inside Π by γ. Counting the total number of planes in a solid, the incident pairs (p,α) where p is a point of K and α a plane, and the number of ordered triples (p,r,α) where p and r are distinct points of K lying in the plane α respectively, yields the following equations,

$$a+b+c = (q+1)(q^2+1),$$

$$a+b(q+1)+c(2q+1) = \gamma(q^2+q+1),$$

$$bq(q+1)+c2q(2q+1) = \gamma(\gamma-1)(q+1).$$

We can calculate a, b and c exactly for each value of γ; later on we will only use that $c=0$ if $\gamma = q^2+1$, that a, b and c are all non-zero if $\gamma = q^2+q+1$, and that $a=0$ if $\gamma = q^2+2q+1$.
Note that it never occurs that two of the integers a, b and c are zero.

Lemma 2.1 *Small solids intersect K in an ovoid.*

Proof Consider a small solid Π and all planes through a line L inside Π, where we assume that L contains $x \geq 2$ points of K. Since a small solid contains no large planes we get exactly

$$(q+1)(q+1-x)+x = q^2+1$$

points, hence $x = 2$. For $q = 2$ we have 5 points, no four coplanar. So for all q, small solids intersect K in an ovoid. ∎

We first prove that the size assumption of theorem 1.9 is fulfilled.

Lemma 2.2 *The set K contains q^3+q^2+q+1 or q^3+q^2+2q+1 points.*

Proof 1) If a small plane α exists, then consider all solids through α inside the 4-dimensional space Δ. We obtain the following lower bound on the size of K,

$$|K| \geq 1+(q+1)q^2 = q^3+q^2+1.$$

Equality holds if and only if all solids through α are small, and small solids are ovoids. Take a line L inside Δ. If L lies in a solid through α, then L contains at most 2 points of K.
Next consider a line M not intersecting α and assume it contains a point x of K. Consider the small solid Π spanned by x and α. Inside Π one can find a small plane containing x. Hence, M lies in a small solid through a small plane, a case already treated. So all lines intersect K in at most 2 points.
Hence, we would find a cap of size q^3+q^2+1. This yields a contradiction with lemma 1.7.
So at least one solid through α is medium or large, so $|K| \geq q^3+q^2+q+1$. In both cases, there is a large plane. Let π be this large plane. Look at all solids through π inside Δ. We get the inequality,

$$|K| \leq (q+1)q^2+2q+1 = q^3+q^2+2q+1.$$

2) If no small plane exists, then all 3-spaces are large ones. In this case, we get the following size for K:

$$|K| = (q+1)q^2+2q+1 = q^3+q^2+2q+1.$$

Taking an arbitrary plane and looking at all solids through it learns that the number of points in K is always 1 mod q, hence this lemma is proved. ∎

Lemma 2.3 *There exist small and medium solids.*

Proof We show that for both possible values of $|K|$, there exist small and medium solids. Denote the number of small, medium and large solids in the 4-dimensional space by a, b and c respectively.
Counting the total number of solids Π in a 4-dimensional space, the number of incident

pairs (p,Π) where $p \in K$, and the number of ordered triples (p,r,Π) where p and r are distinct points of K incident with Π, yields the following equations,

$$a+b+c=\frac{q^5-1}{q-1},$$

$$(q^2+1)a+(q^2+q+1)b+(q^2+2q+1)c=|K|\frac{q^4-1}{q-1},$$

$$(q^2+1)q^2a+(q^2+q+1)(q^2+q)b+(q^2+2q+1)(q^2+2q)c=|K|(|K|-1)\frac{q^3-1}{q-1}.$$

Solving these equations yields that in both cases $a \neq 0$ and $b \neq 0$, so there exist small and medium solids. ∎

In our previous lemmas we have proved all the necessary conditions for theorem 1.9, hence we have the following result.

Theorem 2.4 *If a set of points K in $PG(4,q)$ is such that it intersects all planes in 1, $q+1$, or $2q+1$ points and all solids in q^2+1, q^2+q+1 or q^2+2q+1 points, then it is a parabolic quadric $Q(4,q)$.*

3 The characterization

Consider a set of points K in $PG(n,q)$, $n \geq 4$, that has as intersection numbers with planes

$$1,\ q+1,\ 2q+1,\ q^2+q+1$$

and as intersection numbers with solids

$$q+1,\ q^2+1,\ q^2+q+1,\ q^2+2q+1,\ 2q^2+q+1,\ q^3+q^2+q+1.$$

We adopt the following terminology for the rest of this paper. We call planes and solids that intersect the set K in i and j points respectively, i-planes and j-solids respectively. A line containing $q+1$ points of the set K is called a full line, a (q^2+q+1)-plane will be called a full plane, and a (q^3+q^2+q+1)-solid will be called a full solid.

Lemma 3.1 *A $(2q^2+q+1)$-solid meets the set K in the union of two full planes.*

Proof Consider a $(2q^2+q+1)$-solid Π, a line L contained in Π and look at all planes through L inside Π. Suppose that L contains x points of the set K. Then, if we suppose that Π does not contain a full plane, we find at most

$$(q+1)(2q+1-x)+x$$

points. We find that $x \leq 2$, but then we would have a cap of size $2q^2+q+1$ in $PG(3,q)$. This is impossible, hence Π does contain a full plane, say π. Next consider a point p in $\Pi\backslash\pi$ belonging to $\Pi \cap K$, and let L be a line through p in Π such that L does not lie in a full plane of Π; hence L lies only in $(2q+1)$-planes of Π. Call x the number of points in $K \cap L$. Then we get the following equality,

$$x + (q+1)(2q+1-x) = 2q^2 + q + 1.$$

Hence, $x = 2$. If there is no full plane through p in Π, this would mean that $K = \Pi \cup \{p\}$, which is a contradiction. Hence, we have shown that Π meets K in the union of two full planes. ∎

Lemma 3.2 *A $(q+1)$-solid meets K in a full line.*

Proof Since by assumption every plane is blocked, and since a $(q+1)$-solid contains only $q+1$ points of K, the proof is finished by theorem 1.4. ∎

Lemma 3.3 *If a solid Π contains a full plane π and a point $p \in K\backslash\pi$, then Π is a $(2q^2+q+1)$-solid or a full solid.*

Proof Since Π already contains q^2+q+2 points of K, we only have to prove that Π is not a (q^2+2q+1)-solid. Suppose it is a q^2+2q+1-solid. Consider a line N through p inside Π intersecting K in x points. Consider all planes through N inside Π. They all intersect K in at least $q+2$ points and hence in at least $2q+1$ points. Counting yields the following equality,

$$(q+1)(2q+1-x) + x = q^2 + 2q + 1.$$

This is only possible if $x = q+1$. Since N was an arbitrary line through p in Π, Π would intersect K in more than q^2+2q+1 points, a contradiction. ∎

Lemma 3.4 *There exist full lines.*

Proof If there exists a full plane or a $(q+1)$-solid, then we are done. So suppose that these do not exist. Then by the previous lemmas, there is a 4-dimensional space Δ whose planes are only 1-planes, $(q+1)$-planes and $(2q+1)$-planes and whose solids are only (q^2+1)-solids, (q^2+q+1)-solids and (q^2+2q+1)-solids. But then by theorem 2.4, Δ meets K in a parabolic quadric $Q(4,q)$; which contains lines. ∎

We define a point-line geometry $S = (P, B, I)$, where the points of P are the points of K, where the lines of B are the full lines and where incidence is containment.

Theorem 3.5 *The geometry S is a Shult space.*

Proof We have already shown that there exist full lines, so B is non-empty. The different cases we consider in this proof will also show that B contains at least two lines.

Consider a point p of S and a line L of S, such that p and L are not incident. We prove the axiom for the incidence relation of a Shult space, and we refer to it as the 1-or-all axiom (see page 2 for the definition of a Shult space).

Consider the plane α generated by p and L. Since this plane contains at least $q+2$ points of S, it is either a $(2q+1)$-plane or a full plane. If this plane is a full plane, then we have the all part of the 1-or-all axiom.

So suppose from now on that α is a $(2q+1)$-plane. We distinguish several cases that cover all possible situations.

1) Suppose that there exists a solid Π through α containing a full plane β. If Π was a full solid, then α would be a full plane. So Π either is a (q^2+2q+1)-solid or a $(2q^2+q+1)$-solid.

Since $\beta \neq \alpha$, lemma 3.3 shows that Π is a $(2q^2+q+1)$-solid. By lemma 3.1, Π contains two full planes, they both intersect α in a line, hence the 1-axiom is fulfilled.

2) Suppose now that there exists a 4-space Δ containing α that does not contain full planes. Since $(2q^2+q+1)$-solids and full solids contain full planes, also these do not occur in this 4-space.

a) Suppose that also no $(q+1)$-solids occur in Δ. Then we have exactly the intersection numbers with planes and solids as required for theorem 2.4, so that S intersects Δ in a parabolic quadric $Q(4,q)$, which is a generalized quadrangle, so we have proved the 1-axiom.

b) Suppose that a $(q+1)$-solid Π does occur in Δ, and that it intersects S in a full line M different from L. Consider all planes through M in Δ. Then we find at most

$$q^2(2q+1-(q+1))+(q+1)=q^3+q+1$$

points of S in Δ. Consider all lines through p inside α. One of them, say N, intersects S in exactly 2 points, otherwise α would intersect S in more than $2q+1$ points. Consider all planes through N inside Δ. Since α is a $(2q+1)$-plane, we find at least

$$(q^2+q)(q+1-2)+(2q+1-2)+2=q^3+q+1$$

points. Comparing these inequalities yields that all planes of Δ containing M and not contained in Π are $(2q+1)$-planes. Hence, all solids of Δ different from Π, intersecting Π in a plane that contains M, contain

$$q((2q+1)-(q+1))+q+1=q^2+q+1$$

points of S. The line L and the solid Π intersect in a point r. Then $\{r\}=L\cap M$. If M lies in α, then we have proved the 1-axiom, so suppose that M does not lie in α. Consider the solid Γ generated by α and M. This solid contains at least two lines, namely L and M, it intersects K in q^2+q+1 points and all planes of this solid are 1-planes, $(q+1)$-planes or $(2q+1)$-planes. Theorem 1.8 gives that Γ intersects K in a cone with as vertex r and base an oval. Hence, the line pr is the only line of S through p intersecting L. We have proved the 1-axiom.

c) The remaining case is that the $(q+1)$-solids in Δ intersect K exactly in L. Let Π be such a $(q+1)$-solid of Δ through L. Considering all planes through L inside Δ yields as above that K intersects Δ in at most q^3+q+1 points. Consider a 1-plane β contained in Π, and consider all solids through β in Δ. Since we know the $(q+1)$-solids contained in Δ contain L, we get at least

$$q(q^2)+q+1 = q^3+q+1$$

points. Considering the two inequalities above learns us that they must be two equalities, so there pass q (q^2+1)-solids through β inside Δ. By lemma 2.1, all (q^2+1)-solids through β inside Δ intersect K in an ovoid of $PG(3,q)$.
We consider the union of all these ovoids and add one extra point of L; hence we have found a cap of size $q(q^2+1-1)+2 = q^3+2$ in $PG(4,q)$, yielding a contradiction with lemma 1.7.
Indeed, take any line N lying in Δ and not in Π. There always exists a solid Π' through N in Δ such that $\beta = \Pi' \cap \Pi$ intersects L in a point. So Π' intersects K in an ovoid and hence N intersects K in at most 2 points.
3) Consider now a 4-space Δ containing α such that no solid through α inside Δ contains a full plane, but Δ does. Call this full plane π.
a) Suppose that $p \in \pi$. Then L does not intersect π. Take a point r on L and consider the solid Π'_r generated by r and π. By lemma 3.3, Π' is a $(2q^2+q+1)$-solid or a full solid.
If a solid Π'_r is a full solid, then r is collinear with p in S. Since α is a $(2q+1)$-plane, we have proved the 1-axiom. Suppose now that all solids Π'_r are $(2q^2+q+1)$-solids. If the full plane of Π'_r through r intersects π in a line through p, then we have again proved the 1-axiom. Suppose that this never happens. Then all the lines pr, $r \in L$, contain only two points of S, namely p and r. But then α contains exactly $q+2$ points of S, a contradiction.
b) Suppose that $p \notin \pi$ and look at the solid generated by the point p and the plane π, call it Π.
Suppose that Π is a full solid. Then it does not contain α. It intersects α in a line of S, hence we have proved the 1-axiom.
If Π is a $(2q^2+q+1)$-solid, then it intersects K in a union of two full planes. But then one of these planes contains p, and we are again in case 3)(a). ∎

Theorem 3.6 *If S is non-degenerate, then it is a non-singular quadric in $PG(n,q), n \geq 4$.*

Proof If there exists a full plane, then S is a non-degenerate Shult space of finite rank at least 3, and since all lines contain at least three points by definition, S with all its subspaces is a polar space. By theorem 1.1 it is a finite classical polar space and by looking at the intersection numbers, we see that S is a non-singular quadric.
If there exists no full plane, then the previous arguments show we have proved for S axiom (GQ3) for generalized quadrangles. Clearly, there is a point p through which there pass two lines of S. Hence, S is a generalized quadrangle.
By theorem 1.3, it is a classical one; going through the list of classical generalized

quadrangles yields it is the non-singular parabolic quadric $Q(4,q)$ or the non-singular elliptic quadric $Q^-(5,q)$. ∎

Suppose now that S is degenerate, so there exist points collinear with all other points. We call such points singular points.

Lemma 3.7 *The singular points of S form a subspace Π_k of $PG(n,q)$.*

Proof Take two singular points p and r of S and consider a point t lying on the line $L = pr$. Surely, $t \in S$. All points on S are collinear with t. Take a point s of S not lying on L and consider the plane generated by s and L. This plane has to be a full one, hence s is collinear with t. ∎

Lemma 3.8 *If S contains singular points, then all lines not intersecting the subspace Π_k formed by the singular points, intersect S in 0, 1, 2 or $q+1$ points.*

Proof Consider a line L not intersecting Π_k. Take a singular point p and consider the plane generated by p and L. Since this plane contains either 1, $q+1$, $2q+1$ or q^2+q+1 points of S by assumption, the statement is proved. ∎

Lemma 3.9 *If $n-k-1 \geq 4$, then S is a cone with vertex a k-dimensional space and base a non-singular quadric.*

Proof If S is degenerate, then look at a complementary space $PG(n-k-1,q)$ of the space Π_k. By assumption, this space does not contain singular points of S. If $n-k-1 \geq 4$, then theorem 3.6 shows that S intersects this space in a non-singular quadric, hence S is a cone with vertex a k-dimensional space and base a non-singular quadric. ∎

Now we consider all other cases one by one.
a) If $n-k-1=-1$, then S is the projective space $PG(n,q)$.
b) If $n-k-1=0$, then S is a hyperplane of $PG(n,q)$.
c) If $n-k-1=1$, then the complementary space is a line. If this line intersects K in zero points, we have an $(n-2)$-dimensional space. If it intersects K in 2 points, we have the union of two hyperplanes.
d) If $n-k-1=2$, then the complementary space is a plane π. Suppose that π intersects S in $q+1$ points. Since all lines intersect $K \cap \pi$ in 0, 1, 2 or $q+1$ points, the intersection of π and S is an oval (a line is impossible otherwise we have extra singular points). Suppose that π intersects K in $2q+1$ points. Since π contains more than $q+2$ points of K, π surely contains a line L of S. Take a point $p \in S \cap \pi$ outside L. Considering all lines through p in π learns that one of them is a line of S. The intersection of the two lines would be a singular point, this yields a contradiction.
e) If $n-k-1=3$, then the complementary space is a solid Π.
If this solid intersects S in q^2+1 points, it intersects S in an ovoid.

If this solid intersects S in $q^2 + q + 1$ points, it surely contains a line L of S. Take a point p on S, $p \notin L$, inside Π. Then the plane generated by p and L intersects S in two lines, as before. Hence, Π contains at least two lines. Theorem 1.8 learns that S intersects Π in a cone with vertex a point p and base an oval. This yields a contradiction, since the point p is then a singular point of S.
Suppose Π intersects S in q^2+2q+1 points. By lemma 3.3, we may assume Π intersects all planes in 1, $q+1$ or $2q+1$ points. Again we surely have lines of S lying in $S \cap \pi$. Consider a point p of $S \cap \pi$ and a line L of S, with $p \notin L$. The plane α generated by them is a $(2q+1)$-plane and the intersection sizes of lines immediately prove axiom (GQ3) for generalized quadrangles. By assumption, there is no point of S in Π collinear with all other points of $\Pi \cap S$.
So $S \cap \pi$ is a generalized quadrangle. Again by theorem 1.3, it is a classical one and hence it is $Q^+(3, q)$.
If Π intersects S in $2q^2+q+1$ points, then, by lemma 3.1, we get extra singular points, this yields a contradiction.
Now we can state our main theorem.

Theorem 3.10 *If a set of points K in $PG(n, q)$, $n \geq 4$, intersects planes and solids in the same number of points as quadrics, then K is either*

(i) The projective space $PG(n, q)$,

(ii) A hyperplane in $PG(n, q)$,

(iii) A quadric in $PG(n, q)$.

(iv) If q is even, it can also be a

(iv.1) A cone with vertex an $(n-3)$-dimensional space and base an oval.

(iv.2) A cone with vertex an $(n-4)$-dimensional space and base an ovoid.

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